curvature of the meridian ellipse and r_1 for the radius of the parallel circle and the earth is assumed to be an ellipsoid of revolution. This shows that the Bonne projection is one of equivalence.

From the analytics of the meridian ellipse it follows that,

$$\rho = \frac{a(1-e^2)}{\sqrt{(1-e^2\sin^2\phi)^3}}, \qquad r_1 = \frac{a\cos\phi}{\sqrt{1-e^2\sin^2\phi}},$$

where $2a$ is the major axis, and e the numerical eccentricity of the meridian ellipse.

The element of area will therefore be,

$$\frac{a^2(1-e^2)\sin\chi\, d\chi\cdot d\lambda}{(1-e^2\cos^2\chi)^2},$$

where $\chi = 90^\circ - \phi$ is introduced in order that R and χ may both be increasing functions.

In order to compute the radius R of a parallel circle of the map we start from R_m, the radius of the middle parallel,

$$R_m = r_{1m}\operatorname{cosec}\phi_m = \frac{a\operatorname{ctg}\phi_m}{\sqrt{1-e^2\sin^2\phi_m}}$$

and obtain

$$R = R_m - \int_{\phi_m}^{\phi}\rho d\phi = R_m + \int_{\chi_m}^{\chi}\frac{a(1-e^2)}{\sqrt{(1-e^2\cos^2\chi)^3}}d\chi.$$

The polar angle L is derived from $RdL = r_1 d\lambda$ or

$$dL = \frac{a\cos\phi\cdot d\lambda}{R\sqrt{1-e^2\sin^2\phi}};$$

since R is only a function of ϕ and not of λ, we conclude,

$$L = \frac{a\cos\phi\cdot\lambda}{R\sqrt{1-e^2\sin^2\phi}}, \tag{1}$$

if we count the longitudes and angles L from the primary meridian. Knowing R and L for given values of ϕ and λ, the map is easily constructed. We now turn our attention to the distortions of the map in angle and distance. We shall see that these distortions are zero for the principal meridian and the middle parallel and grow rather rapidly as the point to be represented increases in longitude and approaches the northern or southern edge of the map. To show this we consider first the law of divergence ψ of the direction of radius R of the map from the particular meridian at a point (Fig. 1). It should be remembered that L is not the longitude of the point P on the map since VP is not a meridian in the Bonne projection. In order to obtain the "convergence of meridians" with respect to the primary meridian it is necessary to find the angle ψ which the

tangent to the meridian at P makes with R:

$$\operatorname{tg}\psi = \frac{R\dfrac{\partial L}{\partial \phi}\cdot d\phi}{dR};$$

since

$$R\cdot L = \frac{a\cos\phi}{\sqrt{1-e^2\sin^2\phi}}\cdot\lambda,$$

we obtain

$$R\frac{\partial L}{\partial \phi}\cdot d\phi + L dR = -\frac{a(1-e^2)\lambda\sin\phi d\phi}{\sqrt{(1-e^2\sin^2\phi)^3}} = -\rho d\phi\cdot\lambda\sin\phi = dR\cdot\lambda\sin\phi.$$

The equation for $\operatorname{tg}\psi$ becomes, therefore,

$$\operatorname{tg}\psi = \lambda\sin\phi - L = \lambda\left(\sin\phi - \frac{a\cos\phi}{R\sqrt{1-e^2\sin^2\phi}}\right). \tag{2}$$

ψ will be zero for two cases, (1) $\lambda = 0$, *i. e.*, on the primary meridian, (2) on the middle parallel, since

$$\sin\phi_m = \frac{a\cos\phi_m}{R_m\sqrt{1-e^2\sin^2\phi_m}}.$$

The angle ψ for points east of the prime meridian will be in the NW. quadrant

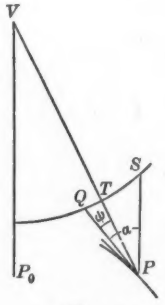

FIG. 1.

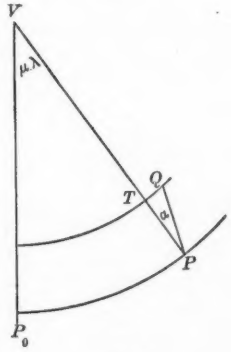

FIG. 2.

(with respect to VP) for points north of the middle parallel, and in the NE. quadrant south of this parallel.[1] If α and a are corresponding azimuths on the

[1] The convergence of meridians with respect to the primary meridian is the angle which the tangent to the meridian at P forms with the primary meridian. This convergence M is measured by $M = L + \psi$ or $= L +$ arc $\operatorname{tg}(\lambda\sin\varphi - L)$, or $\operatorname{tg}(M - L) = \lambda\sin\varphi - L$; for small values of ψ we have $M = \lambda\sin\varphi\cdots$. (In Major W. C. Clark's pamphlet *Heavy Artillery (Coast) Orientation*, p. 38, it is overlooked that M is a function of ψ.)

earth and the map, then $\alpha - a$ will be *the distortion in angle,* and we have

$$\text{tg}\,(a-\psi)=\frac{QS-QT}{PT} \qquad \text{or} \qquad \text{tg}\,(a-\psi)=\text{tg}\,\alpha-\text{tg}\,\psi,$$

or

$$\text{tg}\,\alpha=\frac{\sin a}{\cos\,(a-\psi)\cos\psi}.$$

For the middle parallel ($\psi = 0$) we obtain $\alpha = a$.

The distortion $\alpha - a$ will be found by

$$\text{tg}\,(\alpha-a)=\frac{\text{tg}\,a\,[\cos a-\cos\psi\cos\,(a-\psi)]}{\cos\,(a-\psi)\cos\psi+\sin a\,\text{tg}\,a}. \tag{3}$$

For $\psi = 0$ we have $\alpha = a$ as it should be. Likewise $\alpha = a$ for $tga = 0$ i. e., for the meridian.

To find the *distortion in distance* δ we have on the ellipsoid

$$\Delta\delta=\frac{a(1-e^2)\Delta\phi}{\sqrt{(1-e^2\sin^2\phi)^3}}\cdot\frac{1}{\cos\alpha},$$

and on the map

$$\Delta d=\frac{a(1-e^2)\Delta\phi}{\sqrt{1-e^2\sin^2\phi}}\cdot\frac{1}{\cos\,(a-\psi)}. \tag{4}$$

Hence

$$\frac{\Delta\delta-\Delta d}{\Delta d}=\frac{\cos\,(a-\psi)-\cos\alpha}{\cos\alpha},$$

the distortion $\Delta\delta - \Delta d$ is then likewise a function of ψ and therefore of λ and ϕ, and it increases considerably with the longitude.

The Lambert Conical Conform Projection.[1]

While in the Bonne projection the map-cone is tangential to the earth's surface at the middle parallel of the zone so that the radius of the middle map-parallel is a fundamental constant, the Lambert projection retains the properties of the cone without fixing its position from the start. Since meridians and parallel circles on the earth are orthogonal to each other a conical map projection which is to be *conform* will have to retain this orthogonal system of coördinates. Calling α the initial azimuth of a course on the surface of the earth and a the corresponding angle on the map, then for conform representation $\alpha = a$. Since $\cot\alpha = -\rho d\phi/r_1 d\lambda$, $-\rho d\phi$ being an element of the meridian arc, and $r_1 d\lambda$ that of the parallel circle, and since furthermore $\cot a = dR/RdL$, therefore $dR/RdL = \rho d\phi/r_1 d\lambda$, Fig. 2, L being here the "convergence of meridians."

We may simplify this differential equation considerably by making R only a

[1] Attention is drawn to Special Publications nos. 52 and 53 (price 25 and 10 cents respectively), of the U. S. Coast and Geodetic Survey, by Oscar S. Adams. They were issued by the Government Printing Office, Washington, in 1918, and have the following titles: *Lambert Projection Tables for the United States,* and *General Theory of the Lambert Conformal Conic Projection*—Editor.

function of ϕ and by *assuming* $dL = \mu d\lambda$, where μ is an arbitrary constant, to be determined later by special conditions.

Then we get

$$\frac{dR}{R} = \frac{-\mu\rho d\phi}{r_1} = \frac{-\mu(1-e^2)d\phi}{\cos\phi\,(1-e^2\sin^2\phi)}.$$

Integrating,

$$lg(R) = \mu lg\left[\operatorname{tg}\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)^{e/2}K\right],$$

K being the constant of integration. This gives

$$R = C\cdot\operatorname{tg}^{\mu}\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)^{\mu e/2}, \tag{5}$$

where $C = K^{\mu}$. If we put

$$\operatorname{tg}\frac{\zeta}{2} = \operatorname{tg}\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)^{e/2},$$

ζ will be determined to a sufficient degree of approximation (within a few tenths of a second) on setting

$$\operatorname{tg}\frac{\zeta}{2} = \operatorname{tg}\left(\frac{\pi}{4}-\frac{\phi}{2}\right).$$

Then R takes the final form

$$R = C\operatorname{tg}^{\mu}\zeta/2. \tag{6}$$

To determine the arbitrary constants μ and C, we assume that the map-cone cuts the earth's surface along the parallel circles whose latitudes are ϕ' and ϕ''. These are generally selected close to the northern and southern edges of the territory to be mapped. For the war maps it was desired to have the parallels of 52 grades (Central Switzerland) and 58 grades (Northern Holland) as extreme parallels. By taking $\phi' = 53$ grades and $\phi'' = 57$ grades the determination of the constants was effected for the war maps. To obtain μ, Lambert postulated that

$$\frac{R'}{R''} = \frac{r_1'}{r_1''};$$

this gives

$$\left(\frac{\operatorname{tg}\zeta'/2}{\operatorname{tg}\zeta''/2}\right)^{\mu} = \frac{\cos\phi'\sqrt{1-e^2\sin^2\phi''}}{\cos\phi''\sqrt{1-e^2\sin^2\phi'}},$$

or taking logarithms

$$\mu = \frac{lg\dfrac{\cos\phi'}{\cos\phi''}\dfrac{\sqrt{1-e^2\sin^2\phi''}}{\sqrt{1-e^2\sin^2\phi'}}}{lg\left(\dfrac{\operatorname{tg}\zeta'/2}{\operatorname{tg}\zeta''/2}\right)}. \tag{7}$$

In order to find C the condition is imposed that the linear distortion $k-1$ for the

Omit lines 3-6 after formula (5).

parallel of latitude ϕ' be zero; it is observed that, owing to the way μ has been found, the distortion on the parallel of ϕ'' will be zero likewise. Since

$$k = \frac{RdL}{r_1 d\lambda} = \frac{\mu R d\lambda}{r_1 d\lambda} = \mu \cdot \frac{R}{r_1},$$

we obtain

$$C = \frac{a \cos \phi'}{\mu \operatorname{tg}^{\mu} \zeta' \sqrt{1 - e^2 \sin^2 \phi'}}.$$

Since the cone is inside the earth's surface between the parallels ϕ' and ϕ'' and outside for the border area of the map, it follows that distances on the map will be smaller than corresponding distances on the earth's surface for the zone between ϕ' and ϕ'' and larger in the border-area. Computation will show that the maximum distortion amounts to 1 : 2037 or 5 : 10000; this, even for ranges of 10 km., will be a negligible quantity.

While geodetic lines on the surface of the earth are not represented by straight lines on the map but appear as slightly curved traces, the deviation for a distance of 100 km. will introduce an angular error in azimuth of less than one minute of arc.

In concluding this paper the reader's attention is once more called to Major Clark's able pamphlet which should prove a most interesting guide for students of surveying and cartography. Gretchel's *Lehrbuch der Karten-Projectionen* has been used in preparing this paper.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

32. In a discussion of the Peaucellier[1] cell by analytic methods the following equations are obtained:

$$(1)\quad (x_2 - x_1)^2 + (y_2 - y_1)^2 - b^2 = 0; \qquad (2)\quad (x_3 - x_1)^2 + (y_3 - y_1)^2 - b^2 = 0;$$

$$(3)\quad (x_2 - X)^2 + (y_2 - Y)^2 - b^2 = 0; \qquad (4)\quad (x_3 - X)^2 + (y_3 - Y)^2 - b^2 = 0;$$

$$(5)\quad x_2^2 + y_2^2 - K^2 = 0; \qquad (6)\quad x_3^2 + y_3^2 - K^2 = 0;$$

$$(7)\quad x_1^2 + y_1^2 - 2cx_1 = 0.$$

The result of eliminating $x_1, y_1, x_2, y_2, x_3, y_3$ gives an equation of the first degree, which establishes that the linkage will trace a straight line. There are various ways of effecting this elimination.

1. What element of the situation is left unused by the following procedure in the elimination?
(*a*) From equations (1), (3), (5) eliminate x_2 and y_2 and obtain an equation

$$(8)\quad f_1(x_1, y_1) = 0.$$

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[1] In the accompanying figure, taken from the article on "Linkages" in the December, 1915, MONTHLY, by Mr. Leavens, the coördinates of the points of the linkage are: O $(0, 0)$; C $(c, 0)$; P_1 (x_1, y_1); M (x_2, y_2); M' (x_3, y_3); P_2 (X, Y).

(*b*) From equations (2), (4), (6) eliminate x_3 and y_3 and obtain an equation

$$(9) \quad f_2(x_1, y_1) = 0.$$

(*c*) From equations (7), (8), (9) eliminate x_1 and y_1 and obtain the desired equation.

2. How should this procedure be supplemented to secure the result?

REPLY BY J. K. WHITTEMORE, Yale University.

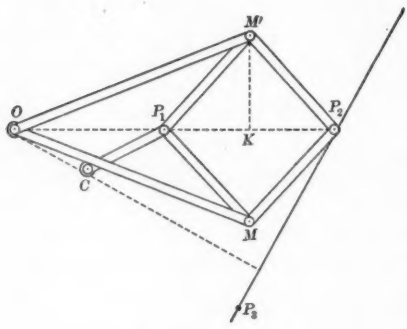
The method of elimination proposed might be expected to fail, since (8) and (9) are apparently identical, for equations (2), (4), (6) differ from (1), (3), (5) only in containing x_3, y_3 in place of x_2, y_2. But the functions f_1 and f_2 of (8) and (9) do not, as a matter of fact, exist at all. If we seek to determine f_1 we may write from (1) and (5)

$$2x_1x_2 + 2y_1y_2 = K^2 - b^2 + x_1^2 + y_1^2,$$

and from (3) and (5)

$$2Xx_2 + 2Yy_2 = K^2 - b^2 + X^2 + Y^2.$$

These equations may be formally solved for x_2, y_2 and the results substituted in (5), apparently giving f_1; but this work is illusory. Equations (2), (4), (6) give identically the same expressions for x_3, y_3, so that, unless (x_2, y_2) coincides with (x_3, y_3), the two equations above are equivalent and cannot be solved, and



$$(A) \quad \frac{x_1}{X} = \frac{y_1}{Y} = \frac{K^2 - b^2 + x_1^2 + y_1^2}{K^2 - b^2 + X^2 + Y^2}.$$

Since (x_2, y_2) and (x_3, y_3) do *not* coincide equations (A) are true. It is indeed obvious from the figure that $Yx_1 - Xy_1 = 0$. The equation of the locus of (X, Y) is obtained by eliminating x_1, y_1 from (A) and (7). The elimination is simply carried out as follows: Let the common value of the fractions in (A) be $\lambda \neq 1$. Then (7) gives

$$\lambda(X^2 + Y^2) - 2cX = 0.$$

Equating to λ the last member of (A),

$$(\lambda^2 - \lambda)(X^2 + Y^2) - (\lambda - 1)(K^2 - b^2) = 0.$$

Since $\lambda - 1 \neq 0$, this factor is cancelled from the last equation; then subtracting the preceding equation,

$$2cX = K^2 - b^2.$$

NOTE. This reply points out the necessity of using the fact that the points (x_2, y_2) and (x_3, y_3) are distinct and that the points (x_1, y_1) and (X, Y) are distinct, the latter condition being of course implied by $\lambda \neq 1$. It is likewise clear from the last part of the proof that neither (x_1, y_1) nor (X, Y) may coincide with $(0, 0)$; this may be insured by requiring that $K \neq b$. There are therefore several elements of the situation left unused by the process suggested in the question. —EDITOR.

DISCUSSIONS.

In a recent number of the MONTHLY, Professor E. J. Moulton considered the nature of a course in advanced calculus which should serve as a bridge between the work in elementary calculus and more advanced subjects. In the first discussion below, Professor Gillespie presents the claims of elementary differential equations for adequately effecting this transition. The content which he proposes is by no means that of the course in elementary differential equations as it is usually given; neither does it coincide with the usual course in advanced calculus. Further expressions of opinion on the questions raised will be welcome.

Among the topics presented in courses common to the secondary and collegiate fields, probably no other is so difficult, and at times so discouraging, to teach satisfactorily, as mathematical induction. Professor Weaver, in the second discussion of this number, gives an amusing account of an analogy from life, which may be useful in illustrating the essential nature of the method of proof involved.

In the third discussion Professor Noble considers the process of altering the strength of a mixture by continuous addition of quantities of one component. Mathematically the situation turns out to involve a neat illustration of the notion of a derivative, leading to a simple differential equation; some of our readers will find here a useful supplementary problem for the class-room.

I. ADVANCED CALCULUS OR DIFFERENTIAL EQUATIONS.

By D. C. GILLESPIE, Cornell University.

The second course in calculus as it is often given,—following Goursat-Hedrick, *Mathematical Analysis*, Vol. I, let us say to be explicit,—breaks suddenly with what has gone before. It does not appeal to everyone as a natural continuation of previous work. The point of view is different, the emphasis is different, and even among those who have done well in the more elementary courses there are students with whom it is not successful. Professor E. J. Moulton's suggestive paper in the December number of the MONTHLY proposes a plan which would accomplish much that it is desirable to accomplish by the second course in the calculus and at the same time make the change less abrupt.

Even in the case of the future mathematicians, Professor Moulton feels, "It is too early in the student's career to introduce the critical attitude of higher mathematics." I am not sure that this phrase is quite clear. The student is perhaps fortunate to escape altogether the attitude that is more concerned with detecting error than with acquiring knowledge. On the other hand the desire and ability to analyze and see on what assumptions theorems are founded and how the results follow from the assumptions are, at every stage of the student's development, great assets. The point is, it seems to me, that while the teacher may encourage this desire he can not introduce this ability. The scales are not going to fall from the student's eyes at the teacher's command; they wear off, and indeed slowly. The plan proposed has rather this clear advantage, that,

instead of halting the progress of the student until he has mastered the fundamental concepts of the calculus, it offers him a course useful and interesting in itself where the ideas of the calculus are continually applied.

Somewhat the same purpose may be accomplished, I think, by an elementary course in differential equations. At most collegiate institutions such a course is given each year. This course may quite naturally cover a good deal of the outline proposed for the second course in calculus.

It is true in my experience with both text books and lectures that courses in differential equations divide themselves rather sharply into two classes: the first is purely formal, where the time is spent in the monotonous and, if long continued, rather deadening work of learning schemes of integration; the second consists in courses devoted to the study of existence theorems and properties of solutions. One must go to books on differential geometry or mechanics or mathematical physics to find the calculus used to state problems and the geometrical and physical notions involved further employed to aid in the solution and in the interpretation of results. It would however be necessary for the purpose at hand to include these applications of calculus under the name elementary differential equations.

The course in differential equations would include methods of integration. It should perhaps include the power series existence theorem for ordinary differential equations of the first order. There could be added many problems from geometry, mechanics and physics which are not stated as differential equations but which the student himself must formulate, solve and interpret. Such a course is a continual review of the ideas of function, continuity, derivative, integral. Families of curves, their trajectories, envelopes and singularities would be included. The plane pendulum problem gives a natural introduction to elliptic integrals and functions. The brachistochrone or some similar problem takes one at least to the Euler equation in the calculus of variations. The existence theorem mentioned or an approximation method leads to some study of Taylor's series and infinite series generally.

The student should acquire in a course of this nature considerable knowledge and technique, as well as a better understanding of the calculus. This would allow more time for the essentially slow process of the accurate statements and proofs of advanced calculus when it follows.

II. Note on the Teaching of the Principle of Mathematical Induction.

By Warren Weaver, University of Wisconsin.

The writer well remembers when he first met a proof by mathematical induction. He felt as it he had been introduced to a scientific three-shell artist. While he could find no definite loop-hole in the argument it seemed too mysterious and unreal a process to have actually effected any definite proof. The attitude of classes meeting this principle for the first time seems to justify the conclusion that this is no unusual circumstance. The fact that some ridiculously simple

illustration from one's everyday experience often wonderfully clears up hazy notions will perhaps justify the following allegory, which has been successful in bringing forth that first, wide-eyed, understanding "oh!" in several cases.

A long—indeed apparently endlessly long—line of people is seen to be standing waiting to buy tickets at a window. One wonders if some far-famed magician is to give an exhibition of extracting square roots from apparently empty silk hats. Our hero, Mr. Kueedee, being the proprietor of a rival attraction, wishes to persuade this line of people to stop waiting, and come to his show. He looks at the people, and seeing a friend, Mr. Kayplusone, he goes over to him.

"Kayplusone," he says earnestly, "I feel that you are a typical average sort of a chap. I come to you to see if I can learn the attitude which all these people take towards the proposition of coming at once to my show."

"I have indeed been thinking that matter over," replies Kayplusone, "and I will agree to come if my friend Kay ahead of me here in line will also go. And I can save you a lot of questions by telling you that mine is a typical attitude. Kay says that he, in turn, will go if the man ahead of him will."

Kueedee ponders for but a moment, and then with sudden inspiration goes to the end of the line, and is seen to be in serious conversation with the man next to the ticket window. Suddenly this man, whose name happens to be One, grabs Kueedee by the hand and calls out, "I'll do it." He speaks hurriedly to the man behind him, and starts off. This man, likewise, speaks to the man behind him, and starts off also. So that Mr. Kueedee goes, calm in the confidence that even the infinite capacity of his house is going to be taxed, and himself puts out the sign,

STANDING ROOM ONLY.

Q. E. D.

III. The Curve of Concentration for a Liquid Mixture.

By C. A. Noble, University of California.

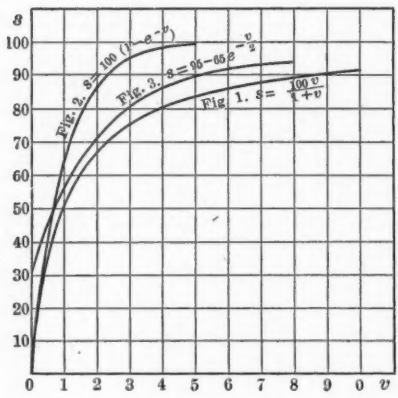
When strong alcohol is added to weak alcohol the concentration of the mixture may be representated by the arc of a hyperbola. If, for example, to a volume v_0, of percentage-strength s_0, is added a volume v, of percentage-strength s_1, the strength of the mixture is given by

$$s = \frac{s_0 v_0 + s_1 v}{v_0 + v}.$$

Figure 1 shows the curve of concentration for the case where pure alcohol is added to a unit volume of pure water. Then $s_0 = 0$, $s_1 = 100$, $v_0 = 1$, and the equation of the curve is

$$s = \frac{100v}{1 + v}.$$

One can obtain more rapidly a desired volume of required concentration if one keeps the volume of the mixture constant, by drawing off from the stirred mixture a volume equal to that of the strong alcohol which has been added. The smaller the steps taken, in this process of adding a small quantity, stirring the mixture, and then pouring off an equal quantity, the more rapidly does the mixture increase in strength. The following discussion is for the case where the process goes on by infinitesimal steps, *i. e.*, continuously.



With the notation mentioned above, let the volume $\Delta v = v/n$ be added to v_0, the mixture stirred, and then the volume Δv of the mixture poured off. The strength of the remaining mixture (volume v_0) is

$$\frac{s_0 v_0 + s_1 \Delta v}{v_0 + \Delta v}.$$

Abbreviate this to k/v_1. Again add Δv of the strong alcohol, stir the mixture, and pour from it Δv. The strength of the mixture is now

$$\frac{s_1 v_1 \Delta v + k v_0}{{v_1}^2}.$$

After another repetition of this process, the mixture has the strength

$$\frac{s_1 \Delta v({v_1}^2 + v_0 v_1) + k {v_0}^2}{{v_1}^3}.$$

After $n - 1$ such repetitions, the mixture has the strength

$$s_n = \frac{s_1 \Delta v({v_1}^{n-1} + v_0 {v_1}^{n-2} + \cdots + {v_0}^{n-2} v_1) + k {v_0}^{n-1}}{{v_1}^n}$$

$$= \frac{s_1 \Delta v}{v_1} \cdot \frac{1 - \left(\frac{v_0}{v_1}\right)^{n-1}}{1 - \left(\frac{v_0}{v_1}\right)} + \frac{k}{v_1}\left(\frac{v_0}{v_1}\right)^{n-1}.$$

If the process is to be continuous, n must become infinite, and the limit s of s_n will be the ultimate concentration. Now,

$$\lim_{n=\infty} \left(\frac{v_0}{v_1}\right)^{n-1} = \lim_{n=\infty} \left[\frac{1}{1 + \frac{v/v_0}{n}}\right]^{n-1} = e^{-v/v_0},$$

and

$$\lim_{n=\infty} \frac{1-\left(\frac{v_0}{v_1}\right)^{n-1}}{1-\frac{v_0}{v_1}} \cdot \Delta v = \frac{v(1-e^{-v/v_0})}{\lim_{n=\infty} n\left[1-\frac{v_0}{v_0+\frac{v}{n}}\right]} = v_0(1-e^{-v/v_0}).$$

Moreover,

$$\lim_{n=\infty} \frac{s_1}{v_1} = \frac{s_1}{v_0},$$

and

$$\lim_{n=\infty} \frac{k}{v_1} = s_0.$$

Inserting these values in the above expression for s_n, one has

$$s = \lim_{n=\infty} s_n = s_1 - (s_1 - s_0)e^{-v/v_0}.$$

This curve has a steeper slope than the hyperbola mentioned above, that is, this method of continuous dilution is the more economical of the two. It is actually employed in biological work at the University of California by Professor Long, who has devised appropriate apparatus, and at whose suggestion the equation of the curve which should represent the process was deduced.

Figure 2 is the curve which represents the pouring of pure alcohol into one unit volume of pure water. $s_1 = 100$, $s_0 = 0$, $v_0 = 1$, and the equation is

$$s = 100(1 - e^{-v}).$$

Figure 3 represents the pouring of 95 per cent. alcohol into two unit volumes of 30 per cent. alcohol. $s_1 = 95$, $s_0 = 30$, $v_0 = 2$, and the equation is

$$s = 95 - 65e^{-v/2}.$$

The differential equation corresponding to this process is simple, and may be deduced as follows:

Let s be the strength of the mixture, the volume of which is maintained at v_0 in the way above indicated. When the small volume Δv of strength s_1 is added, the strength of the mixture changes to

$$\frac{sv_0 + s_1\Delta v}{v_0 + \Delta v}.$$

The increment which this brings to s is

$$\Delta s = \frac{sv_0 + s_1\Delta v}{v_0 + \Delta v} - s = \frac{(s_1 - s)\Delta v}{v_0 + \Delta v}.$$

The difference-quotient becomes

$$\frac{\Delta s}{\Delta v} = \frac{s_1 - s}{v_0 + \Delta v}.$$

Hence, the rate of change of the strength of the mixture with respect to the volume of liquid added, is given by

$$\frac{ds}{dv} = \frac{s_1 - s}{v_0} \quad \text{or} \quad \frac{ds}{s_1 - s} = \frac{dv}{v_0}.$$

The value of v, the volume necessary to raise the strength from s_0 to s, is given by

$$\frac{v}{v_0} = \int_{s_0}^{s} \frac{ds}{s_1 - s} = -\log \frac{s_1 - s}{s_1 - s_0},$$

i. e., by

$$s = s_1 - (s_1 - s_0)e^{-v/v_0}$$

as before.

February, 1919.

RECENT PUBLICATIONS.

REVIEWS.

VEBLEN AND YOUNG'S PROJECTIVE GEOMETRY.

Projective Geometry. By O. VEBLEN and J. W. YOUNG. Boston: Ginn. 8vo. Vol. 1, 1910, reprinted in 1916. 10 + 344 pages; price $4.00. Vol. 2, 1918. 12 + 511 pages; price, $5.00.

To dispel at the outset any misunderstanding concerning the following comment and criticism of the reviewer it must be stated that, in his opinion, Veblen and Young's Projective Geometry is a very scholarly and profound treatise on the axiomatic foundations of projective geometry, its classification, and the main body of projective propositions, or theorems. As such it undoubtedly reflects credit upon American scientific scholarship.

All through the two volumes one is constantly impressed with the fact that the axiomatic foundations are extremely important, and that the chain of principal theorems is merely a byproduct of axiomatic, and not the main object of, scientific research in general. This impression may of course be due to the fact that there are two irreconcilable classes of scientific minds: idealists and realists, just as in philosophy.

For this reason it is as a rule useless to argue a case belonging to the philosophy, or psychology, of this domain between representatives of the two different classes. The idealist maintains that the mind is free to create anything it pleases, without reference to any sense-perception whatsoever; mathematics in its foundation may be based upon pure logic alone. It is the purely logical process which, for him, is the essential thing. The realist, on the other hand, holds fast to the famous doctrine: "Nihil est in intellectu quod non prius fuerit in sensu." According to him our entire knowledge, fundamentally, may be traced back to the impressions made upon our mind by the influx through the senses. The choice

of a set of axioms for a reasonable, not only theoretically possible, scientific doctrine, must be dictated by the desire to create a system whose possible interpretations and applications do not collide in essential parts with our empirical knowledge of the external world. For him the logical structure is erected for the purpose of housing mathematical content conveniently and elegantly. That the foundations of such a system cannot be established without intuitional direction, at least in parts, seems clear to the realist. Poincaré,[1] for example, maintains that some of Hilbert's axioms of order are not free creations of the mind, on the contrary that they bear the mark of intuitional truth, that they are intuitional propositions. According to Poincaré, who, as is well known, has meditated profoundly upon analysis situs, the axiomatic foundations of this important branch of mathematical science cannot be laid entirely without some direction by intuition.

The idealist and realist agree, of course, that after the establishment of a set of axioms the erection of the scientific system, or systems, for which the set is valid, must proceed rigorously according to the demands of formal logic.

But here is another difficulty. How are we going to prove that a system is consistent without the aid of some previously known consistent system? Or are we supposed to go on building the structure and to say that it is consistent as long as it does not collapse; *i. e.*, as long as no contradictions turn up? In this connection see the foot-note on page 9 of volume 1.

After this short digression it would seem apparent that the set of assumptions *A*, *E*, *K* for a real projective space, for example, in spite of the author's assertion, is not "very arbitrary," but is chosen with the purpose of creating a real projective geometry, as it is customarily understood.

Volume I is the product of the joint labor of Veblen and Young, while for Volume 2 Veblen alone holds himself responsible. According to some statements in the preface the authors realize that from the standpoint of pedagogy the contents might possibly be treated and may be arranged differently. "The second volume has been arranged so that one may pass on a first reading from the end of chapter VI, volume 1, to the beginning of volume 2. The later chapters of volume 1 may well be read in connection with the part of volume 2 from chapter V onward."

On page 95, volume 1, the following *provisional assumption of projectivity* is made:

P. *If a projectivity leaves each of three distinct points of a line invariant, it leaves every point of the line invariant.* This is, of course, von Staudt's theorem, to which more explicit reference should be made. Based upon this the fundamental theorem of projective geometry, printed on the same page, then follows easily: *If* 1, 2, 3, 4 *are any four elements of a one-dimensional primitive form, and* 1′, 2′, 3′ *are any of three elements of another or the same one-dimensional primi-*

[1] *Dernières Pensées*, 1917, pp. 92–97. See in this connection also recent discussions in the *Monist* (January, 1919) by Richard A. Arms on "the relation of logic to mathematics" and by V. F. Lenzen on "independence proofs and the theory of implication."

tive form, then for any projectivities giving $1234 \barwedge 1'2'3'4'$ and $1234 \barwedge 1'2'3'4_1'$, *we have* $4' = 4_1'$.

The foregoing provisional assumption contains the key to the whole situation, and its rigorous proof from a set of axioms of connection, order, and continuity has required the successive efforts of von Staudt, Klein, Lüroth, Zeuthen, Darboux, and others.

The discussion of the foundations is again taken up in chapter I of volume 2, with the object of obtaining a classification of various projective geometries. In this the symbols denoting the various assumptions have the following meaning:

A: Alignment,
E: Extension,
P: Projectivity,
K: A geometric number system is isomorphic with the real number system of analysis,
H: If any harmonic sequence exists, not every one contains only a finite number of points,
H_0: The diagonal points of a complete quadrangle are non-collinear.
J: A geometric number system is isomorphic with the complex number system of analysis,
C: Continuity,
R: On at least one line, if there is one, there is not more than one chain.
$\bar{R}$: On some line, l, not all points belong to the same chain.
I: Through a point P of any chain C of the line l, and any point J on l but not in C, there is not more than one chain of l which has no other point than P in common with C,
$\bar{H}$: If any harmonic sequence exists, at least one contains only a finite number of points.
Q: There is more than one net of rationality on a line.
S: Senses of ordered projective spaces. A space satisfying assumptions.
A, E is a general projective space,
A, E, P is a proper projective space,
A, E, H is a non-modular projective space,
$A, E, \bar{H}$ is a modular projective space,
A, E, S is an ordered projective space,
$A, E, \bar{H}, Q$ is a rational modular projective space,
A, E, H, Q is a rational non-modular projective space,
A, E, H, C, R or A, E, K is a real projective space,
$A, E, H, C, \bar{R}, I$ or A, E, J is a complex projective space.

A set of assumptions is said to be categorical, if there is essentially only one system for which the assumptions are valid, i. e., if any two such systems may be made simply isomorphic, page 6, volume 1. Thus A, E, H, C, R form a categorical set.

This classification, the establishment of series of propositions in these various geometries, and the logistic theory leading to them, characterizes the whole

treatise. It is very commendable that the authors have accomplished this task, which was not a very easy one, with marked success. Americans, and for that matter all geometers to whom the book is accessible, will be glad to refer students who desire more information on foundational methods in geometry to Veblen and Young's competent treatment of this branch of mathematics.

As far as the didactic treatment is concerned, the reviewer expresses the opinion that in a projective geometry a more central, or prominent position should be given to the fundamental theorem, and it would have been more consistent to have incorporated the first chapter of volume 2 in volume 1 before the fundamental theorem was reached. Enriques's *Vorlesungen über projektive Geometrie* offers an excellent example of such a treatment. Also Del Pezzo's *Principi di Geometria Projettiva* follows a similar plan with much success.

Although a logistic treatment of geometry should be possible without figures of any kind, Veblen and Young were wise not to exclude the mechanical auxiliary of graphic illustrations, as is done by Del Pezzo. I ask the question whether anybody without preliminary knowledge, or the analytic equivalent, could understand Del Pezzo's book at all without drawing most of the figures to which he refers? Again, who is able to write a treatise on projective geometry without either the knowledge of the analytic equivalent, or the drawing of at least some of the figures as a purely mechanical auxiliary? If an author makes use of figures in preparing a treatise, why should those figures be deleted on publication? But the point is really this: Since a logistic system starts with a set of *undefined elements*, and a set of assumptions between them, independent of intuition, the use of figures in such a system should be justified. What relation connects the points and lines of the logistic system with the physical points and lines of the figure and the sheet of paper upon which they are drawn? It seems to me that this difficulty can only be overcome by a set of *axioms of construction* admitting the possibility of actual graphic representation, isomorphic with the logistic system.

As to the content of the two volumes in general, it would be impossible to give a detailed account of it within the narrow space of this review. I shall therefore confine myself to some features of particular interest.

The fourteen chapters of the first volume deal with theorems of alignment and the principle of duality; projection, section, perspectivity, elementary configurations; projectivities of the primitive geometric forms of one, two, and three dimensions; harmonic constructions and the fundamental theorem of projective geometry; conic section; algebra of points and one-dimensional coördinate systems; coördinate systems in two and three-dimensional forms; geometric constructions, invariants; projective transformations of two-dimensional forms; families of lines.

As has been pointed out before, von Staudt's theorem, and thereby implicitly the fundamental theorem, is assumed for a proper projective geometry and receives its conclusive aspect at the beginning of the second volume. Thus the theory of conics in chapter 5, for example, is based upon this assumption P

(and, of course, A, E, H_0). An account of the historic development, or of the evolution, of the fundamental theorem, and a statement as to the motive for making one particular choice of assumptions in preference to another, would have greatly added to the clearness of the exposition. The reviewer shares with Poincaré the opinion that the beginner should be given the reason why a particular choice of axioms is made, because it seems to be evident that the choice cannot be "very arbitrary" if the geometer has in view the establishment of a certain system of geometry. Moreover the student undoubtedly shows increased interest if he is familiarized with the historic roots of the science which he undertakes to study. The authors, in their treatise at least, show little interest in the historic side of the subject. Historic and some other references given are rather meager and partly insufficient. Thus, in a foot-note on page 109 we are told that Jakob Steiner (1796–1863) was the first to define the first four one-dimensional forms of the second degree by projective pencils. Why not add when and where?, *i. e.*, in the *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander*, etc., which appeared in 1832.

In the definition of homogeneous coördinates in the plane it is important to state explicitly that the ratios of any two coördinates of a point (x_1, x_2, x_3), or a line (u_1, u_2, u_3), may be represented by cross-ratios, which are not changed by collineations. This is the reason why they are frequently called projective coördinates, as introduced into geometry by Fiedler.[1] It would be more natural to incorporate exercise 1, on page 179, into theorem 3 on the same page. Choosing the same triangle of reference for point and line coördinates, and a unit line ($\rho u_1 = 1$, $\rho u_2 = 1$, $\rho u_3 = 1$) independent of the unit point ($\delta x_1 = 1$, $\delta x_2 = 1$, $\delta x_3 = 1$), the condition for coincidence for a point (x_1, x_2, x_3) and a line (u_1, u_2, u_3) is $k_1u_1x_1 + k_2u_2x_2 + k_3u_3x_3 = 0$, where k_1, k_2, k_3 are three constants which do not vanish simultaneously. When the unit-line is the triangle-polar of the unit-point, then $k_1 = k_2 = k_3$ and the necessary and sufficient condition for coincidence reduces to $u_1x_1 + u_2x_2 + u_3x_3 = 0$. A similar remark may be made with respect to projective coördinates in space of three dimensions.

In Chapter IX on geometric constructions is introduced the term *geometric operation*, by which the authors give expression to the feeling that something should be done in passing from an axiomatic to a concrete system. This is precisely the point which I had in view when I wrote of the necessity of suitable axioms of construction. The statement, on pp. 236–237, that there is no familiar mechanical device for drawing lines and planes in space is not correct. Just as one can manufacture a ruler, one can manufacture a *planoid*, *i. e.*, a plane surface, by means of which planes in space may be drawn, or mechanically fixed. Moreover we have Koenig's planigraph,[2] a linkage, by which a plane may be described in analogy with the description of a straight line by Peaucellier's inversor.

On account of the importance of the subject it would have been well if the

[1] *Vierteljahrsschrift der naturforschenden Gesellschaft in Zürich*, Vol. 15, p. 152f. (1870).
[2] *Leçons de Cinématique*, pp. 295–297.

classification of collineations and their group-properties in Chapter X had received a more extensive treatment. The same is true of the discussion of pairs of conics on pp. 287–293. The parametrically linear form which is of importance in this discussion is the pencil of conics and the classification of such pencils. We also miss the statement that, in general, the discriminant of a pencil of conics $\phi + \lambda\psi = 0$, set equal to zero, is a cubic in λ, so that, in general, there are three degenerate conics in a pencil.

The last chapter on families of lines deals with the regulus, its principal properties, and the figures of lines depending on these properties. It is important to know that, from a certain point of view, the problem of projectivity is closely related with the theory of the regulus. In arts. 109 and 110 a very brief treatment of Plücker's line coördinates is given. The fact ought to be mentioned that the coördinates

$$\rho_{ik} = \begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix}, \qquad \phi_{ik} = \begin{vmatrix} u_i & v_k \\ v_i & u_k \end{vmatrix}$$

are commonly known as *homogeneous radial*, and *axial coördinates* of the line, respectively. Considering their great importance in other fields, the linear complex and in particular the null system, and line-congruences deserve fuller consideration than is given to them in the book under review. The authors realized, of course, that in their treatment space did not permit of too extensive accounts of even important specific geometric systems.

The scope of the second volume, written by Veblen, is very clearly stated in the preface:

"We have in mind two principles for the classification of any theorem of geometry: (*a*) the axiomatic basis, or bases, from which it can be derived, or, in other words, the class of spaces in which it can be valid; and (*b*) the group to which it belongs in a given space. . . . Having fixed attention on any particular space, we have a set of groups of transformations to each of which belongs its geometry. For example, in the complex projective plane we find among others, (1) the group of all-continuous one-to-one reciprocal transformations (analysis situs), [why add the word reciprocal?], (2) the group of birational transformations (algebraic geometry), (3) the projective group, (4) the group of non-Euclidean geometry, (5) a sequence of groups connected with Euclidean geometry. . . . The two principles of classification, (*a*) and (*b*), give rise to a double sequence of geometries, most of which are of consequence in present day mathematics. It is the purpose of this book [volume 2] to give an elementary account of the foundations and interrelations of the more important of these geometries (with the notable exception of (2))." Veblen suggests the desirability of other books taking account of this logical structure, but dealing with particular types of figures. I take it for granted that the author does not want to intimate by this that there are no treatises on geometry in existence which are based upon a logical structure. The projective geometry of that three dimensional space which is simply isomorphic with analytic space containing as points all quadruples

(x_1, x_2, x_3, x_3) of real or complex numbers, of which not all four ever vanish simultaneously, and as planes all linear equations $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$ between the coördinates of these points, with the a's as arbitrary constants from the same number system, for example, is well established according to the requirements of a logical system.

The second volume contains nine chapters on foundations, elementary theorems on order, the affine group in the plane, Euclidean plane geometry, ordinal and metric properties of conics, inversion geometry and related topics, affine and Euclidean geometry of three dimensions, non-Euclidean geometries, theorems on sense and separation.

Affinity is given a prominent place in Chapters III and VII, which treat of the affine groups in the plane and in space, respectively. As the systems of affinity in the plane as well as in space express properties invariant under a given group of transformations, they may be called geometries in the sense of Klein and properly specified by the word affine. Considering the fact that affinity is recognized as a geometry of marked importance in a number of recent investigations, the 112 pages reserved for the two chapters prove indeed that the author is aware of this fact. Chapter III is based entirely on assumptions A, E, P, H_0. In fact a large number of theorems depend only on A, E, H_0. The affine group of three dimensions, of which the whole affine plane geometry is a part, in a nonhomogeneous coördinate system in which π_∞ is the singular plane, consists of the set of all projectivities of the form

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{10},$$

$$y' = a_{21}x + a_{22}y + a_{23}z + a_{20},$$

$$z' = a_{31}x + a_{32}y + a_{33}z + a_{30},$$

where

$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0.$$

The set of points not on π_∞ is called a *Euclidean space* and π_∞ is called the *plane at infinity* of this space, which is invariant in the affine group of three dimensions. The corresponding geometry is called *affine geometry*, or simply *affinity of three dimensions*.

To non-Euclidean geometries in Chapter VIII only 31 pages are given. These are approached in the most natural manner by means of *the absolute* as introduced by Cayley and clearly brought out by Klein in its connection with non-Euclidean geometries. I wonder what our friend Halsted will say if he finds that in the references to the history and exposition of parts of non-Euclidean geometry on page 362, his name is omitted?

In comparison to the size of this chapter, the one following on theorems on sense and separation with 116 pages, more than one fifth of the whole book, seems exceptionally large. But this chapter contains much of Veblen's own investigations on this branch of analysis situs.

A very commendable feature of the whole treatise is the large number of exercises, which are either original, or selected from many well-known standard works and monographs. Among them are problems of considerable importance in real projective geometry, which by some authors are included in the main subject matter.

The mechanical make up of the two books is excellent, with the exception of many figures, which, from the standpoint of a connoisseur of graphic arts, are by no means the product of expert draftsmen. Pleasing figures, however, are as much to be desired as pleasing type. In this respect the second volume shows a decided improvement over the first.

May be it is a whim of an æsthetic crank if he misses the same kind of imprint on the backs of the two volumes. Why is that of the first volume gilt, that of the second volume black? But we shall not insist further upon such trivialities, and close this review by thinking with Voltaire: Le secret d'ennuyer est celui de tout dire.

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An Elementary Treatise on Curve Tracing. By P. FROST. Fourth edition, revised by R. J. T. BELL. London, Macmillan, 1918. 8vo. 16 + 210 pp. + 17 plates. Price 12s. 6d.

The first edition of this work appeared in 1872 and reprints were issued as second and third editions in 1892 and 1911 respectively. Concerning the present edition (which is the first revision of the original work) Mr. Bell states that it does not differ from the previous one except in places where alterations were necessary to remove ambiguities or to correct mistakes in analysis and diagrams. A useful classified list of the curves discussed has been added on pages 203–208, and the typography of the new edition is much better than that of the earlier one. Except for 7 types of curves whose equations involve trigonometric functions, all curves analyzed have rational algebraic equations; these include 23 cubics, 69 quartics, 31 quintics, 18 sextics, 9 curves of the seventh degree and 2 of the eighth. As the calculus is nowhere employed in the discussion great ingenuity is displayed in the analyses, and the work as a whole is exceedingly interesting.

In his preface of 1872 (reprinted in the present edition) Frost wrote that it would be difficult to find another subject

"which, with a very limited extent of reading, combines, to the same extent, so many valuable hints of methods of calculations to be employed hereafter, with so much pleasure in its use.

For example, the subject of Graphical Calculations is coming more into use every day, and is applied with success to many difficult problems in Statics, Engineering and Crystallography; hints of this the student will find in the practical solution of divers questions and in the determination of the number of their real roots, which are obtained by graphical methods with great facility.

Again, the methods of successive approximations which are employed in Optics and Astronomy are illustrated in the process of finding asymptotes and approximations to the forms of curves at a finite distance.

The comparison of large and small quantities of different orders of magnitude contains the staple of many of the most important applications of Mathematical Analysis; the Lunar and Planetary Theories depending almost entirely upon such considerations of relative magnitude.

The habit of looking towards an infinite distance, and discussing what takes place there, will render less startling a multitude of conceptions having in them a tendency to produce a feeling of vagueness, such, for instance, as the treatment of the mechanical effect of a couple as synonymous with that of an infinitely small force acting at an infinitely great distance.

As an important point, I would mention the tentative character of the inverse problem in which the form of a curve being given, its equation is to be investigated; the kind of uncertainty which will remain on the mind on account of defective estimation of magnitudes; and the necessity of a selection of what may appear the best of many possible solutions; all this will prepare the student for disappointment which, having perhaps a wrong notion of what is meant by calling mathematics an exact science, he will feel in the conflict of theories by which it is attempted to reconcile the results of experiment in such subjects as Heat, Light, Electricity, and Molecular action generally; for an instance of this I may refer to the battle of philosophers about the direction of vibration of the ether in Plane Polarization."

Contents—Chapter I: Introductory theorems; definitions; tracing by points; symmetry, 1–8. II: Orders of small quantities; forms of parabolic curves near the origin; cusps; tangents to curves; curvature, 9–19. III: Forms of parabolic curves at an infinite distance; examples of tracing curves; trigonometrical curves; illustrations of theory of equations; rules for approximation, 20–37. IV: Forms of curves in the neighbourhood of the origin; simple tangents; direction and amount of curvature; multiple points of two branches; curvature of branches at multiple points; multiple points of higher orders, 38–57. V: Forms of branches whose tangents at the origin are the coördinate axes, 58–67. VI: Asymptotes; points of intersection at an infinite distance; asymptotes parallel to the axes, 68–87. VII: Asymptotes not parallel to the axes; asymptotes to homogeneous curves, 88–106. VIII: Curvilinear asymptotes, 107–116. IX: The analytical triangle [Newton's parallelogram and De Gua's triangle]; properties of the analytical triangle, 117–132. X: Singular points; division into compartments; special curve of the fourth degree, 133–166. XI: Systematic tracing of curves; repeating curves, 167–185. XII: Inverse process; determination of the equation of a given curve, 186–202.

Matrices and Determinoids. By C. E. CULLIS. Cambridge: at the University Press. Royal 8vo. Volume 1, 1913; 12 + 430 pp. Price 21 shillings. Volume 2, 1918; 24 + 555 pp. Price 42 shillings.

Extracts from prefaces: "The present work is an amplification of a course of lectures given for the University of Calcutta in the winter of 1909–10. Its chief feature is that it deals with rectangular matrices and determinoids as distinguished from square matrices and determinants, the determinoid of a rectangular matrix being related to it in the same way as a determinant is related to a square matrix. . . . The first volume contains the most fundamental portions of the theory, and concludes with the solution of any system of linear algebraic equations, which is treated as a special case of the solution of a matrix equation of the first degree. . . . The second volume contains those parts of the theory which naturally precede any investigation of the special properties of functional matrices, i. e., matrices whose elements are rational integral functions of a finite number of variables. It deals almost exclusively with matrices whose elements are constants, which may be arbitrary parameters, and with those transformations of such matrices which are classed as equigradent. It does not however contain all the properties of such matrices. There remain many properties which it will be more convenient to consider after a preliminary study of functional matrices. . . . The following is a list of the books which have had most influence on the work as a whole: Bôcher's *Introduction to Higher Algebra*, Heffter and Koehler's *Lehrbuch der Analytischen Geometrie*, Muth's *Elementarteiler*, Netto's *Vorlesungen über Algebra*, Veronese's *Fondamenti di geometria a piu dimensioni*, Whitehead's *Universal Algebra*. My indebtedness to these and other writers will be more easily recognized in those portions of the work, occurring chiefly in volume 3, which are interpolations in the original scheme." It is expected that the third volume will contain the completion of the theory and "applications to vector analysis and the theory of invariants. The complete exposition was in fact undertaken with a view to these last mentioned applications."

Contents—Volume 1, chapter I: Introduction of rectangular matrices and determinoids, pages 1–21; II: Affects of the elements and derived products of a matrix or determinoid, 22–54; III: Se-

quences and the affects of derived sequences, 55–85; IV: Affects of derived matrices and derived determinoids, 86–104; V: Expansions of a determinoid, 105–152; VI: Properties of a product formed by a chain of matrix factors, 153–208; VII: Determinoid of a product formed by a chain of matrix factors, 209–247; VIII: Matrices of minor determinants, 248–264; IX: Rank of a matrix and connections between the rows of a matrix, 265–298; X: Matrix equations of the first degree, 299–363; XI: Solution of any system of linear algebraic equations, 364–417; Index, 419–430—Volume 2, XII: Compound matrices, 1–36; XIII: Relations between the elements and minor determinants of a matrix, 37–106, 515–520; XIV: Some properties of square matrices, 107–164, 521–530; XV: Ranks of matrix products and matrix factors, 165–227; XVI: Equigradent transformations of a matrix whose elements are constants, 228–308; XVII: Some matrix equations of the second degree, 309–377; XVIII: The extravagances of matrices and of spacelets in homogeneous space, 378–462, 531–534; XIX: The paratomy and orthotomy of two matrices and of two spacelets of homogeneous space, 463–514; Index 535–555.

Graphical and Mechanical Computation. By J. LIPKA. New York, Wiley, 1918. 9 + 264 pp. + 2 scales in pocket. Price $4.00.

Contents—I: Scales and the slide rule, 1–19. II: Network of scales; charts for equations in two and three variables, 20–43. III–V: Nomographic or alignment charts, 44–119. VI: Empirical formulas—non-periodic curves, 120–169. VII: Empirical formulas—periodic curves, 170–208. VIII: Interpolation, 209–223. IX: Approximate integration and differentiation, 224–259.

Extract from the Preface—"This book embodies a course given by the writer for a number of years in the Mathematical Laboratory of the Massachusetts Institute of Technology. It is designed as an aid in the solution of a large number of problems which the engineer, as well as the student of engineering, meets in his work. . . .

"Engineers have recognized for a long time the value of graphical charts in lessening the labor of computation. Among the charts devised none are so rapidly constructed nor so easily read as the charts of the alignment or nomographic type—a type which has been most fully developed by Professor M. d'Ocagne of Paris. Chapters III, IV, and V aim to give a systematic development of the construction of alignment charts; the methods are fully illustrated by charts for a large number of well-known engineering formulas. It is the writer's hope that the simple mathematical treatment employed in these chapters will serve to make the engineering profession more widely acquainted with this time and labor-saving device.[1]

"Many formulas in the engineering sciences are empirical, and the value of many scientific and technical investigations is enhanced by the discovery of the laws connecting the results. . . . Chapter VII considers the case where the data are periodic, as in alternating currents and voltages, sound waves, etc. and gives numerical, graphical, and mechanical methods for determining the constants in the equation.

"When empirical formulas cannot be fitted to the experimental data, these data may still be efficiently handled for purposes of further computation,—interpolation, differentiation, and integration,—by the numerical, graphical, and mechanical methods developed in the last two chapters.

"Numerous illustrative examples are worked throughout the text, and a large number of exercises for the student is given at the end of each chapter. The additional charts at the back of the book will serve as an aid in the construction of alignment charts. Bibliographical references will be found in the footnotes.

"The writer . . . owes the idea of a Mathematical Laboratory to Professor E. T. Whittaker of the University of Edinburgh."

[1] The second edition, revised and corrected, of D'Ocagne's *Calcul graphique et nomographie* (Paris, O. Doin, 1914), contains an interesting and extensive bibliography down to the year 1912 (pages 381–386). Among eighteenth century items are: (1) L. Pouchet, "Arithmétique linéaire" appendix to *Echelles graphiques des nouveaux poids, mesures*, Rouen, 1795; and (2) J. von Segner, "Methodus simplex et universalis omnes omnium æquationum radices detegendi," *Acad. Petrop. Novi Comment.*, tome 7, 1761. E. V. HUNTINGTON briefly discusses nomography on pages 178–185 of his *Handbook of mathematics for engineers* (1918) which is reprinted from L. S. Marks's *Mechanical Engineer's Handbook* (New York, 1916). For an application of nomography to the geological problem of finding faults, see "The faultless faultfinder" by W. S. Weeks and E. V. Huntington, *Engineering and Mining Journal*, August 15, 1914.

Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche. Di F. ENRIQUES. Pubblicate per cura del Dott. O. Chisini. Bologna, Zanichelli. 8vo. Vol. 1, 1915, 16 + 398 pp.; vol. 2, 1918, 713 pp. Price 12 + 30 lire.

Contents—volume 1: Libro I, Introduzione (1. Le equazioni $f(x) = 0$ e i gruppi di punti sulla retta; 2. Interpretazioni fondamentali dell' equazione $f(xy) = 0$: curve e corrispondenze; 3. Nota sul significato dell' espressione "in generale" e sui computi di costanti), 5–153. Libro II, Il principio di corrispondenza e le sue applicazioni (1. Le involuzioni e i gruppi finiti di proiettività sulla retta; 2. Teoria elementare delle curve piane; 3. Nota sulle funzioni algebriche e sulle rappresentazioni reali dell'immaginario), 155–386—Volume 2: Libro III, La teoria elementare delle curve piane basata sulla polarità (1. Polarità e curve covarianti; 2. Il problema delle intersezioni e i caratteri plueckeriana delle curve; 3. La cubica piana; 4. Appendice: realità e continuità; geometria numerativa), 5–321. Liber IV, Le singolarità delle curve algebriche (1. Le singolarità e gli sviluppi in serie di Puiseux; 2. Le singolarità rispetto alle trasformazioni quadratiche; 3. Le singolarità rispetto al calcolo differenziale; 4. Appendice: singolarità delle curve gobbe e delle superficie), 323–686. Indice alfabetico, indice dei capitoli, and errata-corrige e addizioni, 687–713.

The History of Statistics, Their Development and Progress in Many Countries, in Memoirs to commemorate the seventy-fifth anniversary of the American Statistical Association, collected and edited by J. KOREN. New York, Macmillan, 1918. 8vo. 12 + 773 pp. Price $7.50.

Contents—Book I: Historical and commemorative addresses, "The American Statistical Association, 1839–1914," by John Koren, president, pages 3–14; "Seventy-five years of progress in statistics: the outlook for the future" by S. N. D. North. Book II: History and development of official statistics in many countries: Australia by G. H. Knibbs, 53–82; Austria by R. Meyer, 83–122; Belgium by A. Julin, 123–176; Canada by E. H. Godfrey, 177–198; Denmark by A. Jensen, 199–214; France by F. Faure, 215–330; Germany by E. Würzburger, 331–362; Great Britain and Ireland by A. Baines, 363–390; Hungary by L. v. Buday, 391–414; India by A. Baines, 415–426; Netherlands by C. A. V. Stuart, 427–444; Norway by A. N. Kiaer, 445–466; Russia by A. Kaufmann, 467–534; Sweden by E. Arosenius, 535–570; United States: Federal statistics by J. Cummings, State statistics by C. F. Gettemy, 571–739.

NOTES.

Il Pitagora has not been published since the completion of "anno 23" in June, 1918—*Rendiconti del circolo matematico di Palermo* resumed publication in February, 1919; the last number which had appeared previously was volume 42, fasc. 1, January–April, 1917.

The publication of *Mathematical Questions and Solutions in Continuation of the Mathematical Columns of the Educational Times,* which had been issued in monthly parts since January, 1916, was discontinued "for the present," in December, 1918, with the issue of the last number of volume 6. The three series of this publication total 110 volumes. (For further details see this MONTHLY, volume 3, 1896, pages 159–163, and volume 22, 1916, page 100).

On Professor Poul Heegaard's appointment as professor at the University of Christiania he transferred the *Nyt Tidsskrift for Matematik, A* and *B,* of which he had long been editor, to the Mathematical Society of Copenhagen. The publication is now being continued as *Matematisk Tidsskrift, A* and *B;* the first numbers appeared in February, 1919. *A* is edited by I. L. W. JESSEN and O. A. SMITH; *B* by H. BOHR, and T. BONNESEN. *Nyt Tidsskrift, A* and *B,* published 1890–1918, constituted a continuation of *Tidsskrift for Matematik,* 1859–89.

The *Subject Index of the Modern Books added to the Library of the British Museum* 1911–1915 (London, 1918), lists many books and memoirs of mathematical interest under such headings as: Aëronautics; algebra; arithmetic; calculus; functions, mathematical; hydraulics; hydrodynamics; mathematics; mechanics; surveying; tables; and trigonometry. The writings of American authors are well represented.

In *The Carnegie Foundation for the Advancement of Teaching, Thirteenth Annual Report of the President and of the Treasurer* (New York, October, 1918), are brief biographies of CHARLES LEE CRANDALL (1850–1917) and of NATHAN FELLOWES DUPUIS (1836–1917). Professor Crandall was connected with Cornell University most of his life and among books of which he was the author are: *Notes on descriptive geometry* (1888, 1893), *The transition curve* (1893, 1899), and *Textbook on geodesy and least squares* (1907). Professor Dupuis was a Canadian and he taught at Queen's University, his alma mater, for more than forty years. He built several well-known astronomical clocks, a sidereal clock built by him being still in use in the university observatory. In addition to numerous articles for periodicals Professor Dupuis wrote a *Treatise on geometrical optics* (1868); *Geometry of the point, line and circle* (1889); *Principles of algebra* (1893); *Elements of synthetic solid geometry* (1893); *Elements of trigonometry for practical science students* (1902); *Spherical trigonometry and astronomy* (1906); *Descriptive and mechanical astronomy* (1910); and *Measurement of time* (1915).

ARTICLES IN CURRENT PERIODICALS.

ANNALI DI MATEMATICA PURA ED APPLICATA, series 3, volume 28, no. 1, November, 1918: "Ulisse Dini" by L. Bianchi, i–ii. [Dini was born at Pisa in November, 1845, and died in his native city, October 28, 1918. He was professor of mathematics in the University of Pisa for over 50 years, and director of the R. Scuola Normale Superiore in Pisa since 1901. His books which were, for the most part, developments of university lectures, include the following: (1) *Fondamenti per la teoria delle funzioni di variabili reali*, Pisa, 1878; (German edition, 1892); (2) *Sopra le serie di Fourier ed altre le representazioni analitiche per le funzioni di una variabile*, Pisa, 1880; (there is a type-written English translation of this work, by W. B. Ford, in the Library of Harvard University); (3) *Teoria delle funzioni ellitiche* (autographed), 1893; (4) *Sugli sviluppi in serie per la rappresentazione analitica delle funzioni di una variabile reale date arbitrariamente in un certo intervallo*, Pisa, 1911; (5) *Lezioni di analisi infinitesimale*, Pisa, 1909–1915 (a forerunner of this thousand page work was the lithographed *Lezioni* of 1877–78).

A portrait and sketch of Dini are given in *Acta Mathematica, 1882–1912. Table générale des tomes 1–35*, Upsala, 1913].

BOLLETTINO DI BIBLIOGRAFIA E STORIA DELLE SCIENZE MATEMATICHE, volume 20, January–March, 1918: "Notizie su la Facoltà di scienze matematiche della R. Università di Modena" by E. Bortolotti, 1–11; Review by G. Loria of L. Braude's *Les coordonnées intrinsèques. Théorie et application* (Paris, 1914), 12; Review by C. Rosati of E. Ciani's *Il metodo delle coordinate proiettive omogenee nello studio degli enti algebrici* (Pisa, 1915), 13–15; Review by E. E. Levi of G. Fubini's *Lezioni di analisi infinitesimale*. 2a ed. (1916), 15–16; Review by G. Vivanti of U. Dini's *Lezioni di analisi infinitesimale*, Vol. 2: calcolo integrale (Pisa, 1909–15), 17–18; Review by G. Fubini of G. Vivanti's *Equazioni integrali lineari*, (Milano, 1916), 18–20; Review by G. Loria of P. Ruffini's *Opere matematiche*, Vol. 1 (Palermo, 1915) and Euclid's *Il primo libro degli Elementi, testo greco, versione italiana, introduzione e note a cura di* G. Vacca (Firenze, 1916), 20–21; Review by P. Burgatti of G. Colonnetti's *Principi di statica dei solidi elastici* (Pisa, 1916), 22–29; Notizie, 30–32.

L'ENSEIGNEMENT MATHÉMATIQUE, volume 20, no. 1, January, 1918: "L' approximation des fonctions d'une variable réelle" by C. de la Vallée Poussin, 5–29; "Deux recents ouvrages de

Géométrie" [by Darboux and d'Ocagne] by A. Buhl, 30–42; "Remarques sur la construction des courbes gauches avec application a la parabole cubique" by G. Loria, 43–47; "Théorie élémentaire de la toupic gyroscopique" by M. Zack, 47–62; Review by M. Plancherel of R. Fueter's *Synthetische Zahlentheorie* (Leipzig, 1917), 70–71; Review by A. Reymond of A. N. Whitehead's *The Organization of Thought* (London, 1917), 73–76.—No. 2, September, 1918: "Sur les congruences linéaires de cubiques gauches douées d'une seule courbe singulière" by L. Godeaux, 81–89; "Sur la gerbe de cubiques gauches passant par cinq points" by F. Gouseth, 90–93; "Sur les trajectoires d'un mobile soumis à une force centrale et a une resistance de milieu" by C. Cailler, 93–96; "Sur certaines identités vectorielles et leurs interprétations dans la géométrie sphérique et plane" by M. F. Daniëls, 97–122; "Pensée axiomatique" by D. Hilbert, 122–136; Reviews by A. Buhl of E. Borel's *Leçons sur les fonctions monogènes uniformes d'une variable complexe* (Paris, 1917) and L. Lecornu's *Cours de Mécanique* (tome 3) and M. d'Ocagne's *Cours de géométrie pure et appliquée de l'Ecole Polytechnique* (Paris, 1918), 143–144, 147–154; Review by M. Plancherel of E. Landau's *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale* (Leipzig, 1918), 146–147.—No. 3, December: "Notions d'arithmogéométrie" by E. Turrière, 161–174; "Sur la 'variété moyenne' de deux variétés convexes" by G. Tiercy, 175–189; "Contribution à la construction des éléments doubles d'une involution hyperbolique" by F. Redl, 190–193; "Extraction de la racine $n^{\text{ième}}$ d'une nombre réel par approximations successives" by M. T. Béritch, 194–198; "Note sur les permutations" by A. Aubry, 199–215; "Sur la rectification approchée d'un arc de cercle" by A. Pleskot, 215–218; "A propos d'un problème de Lagrange sur la construction des cartes géographiques" by L. Ballif, 219–221; "J. H. Graf (1852–1918)" by L. Crelier, 224–225; Review by H. Fehr of *Scritti matematici offerti ad Enrico D'Ovidio in occasione del suo LXXV genetliaco, 11 agosto, 1918* (Turin, 1918), 229–230.

L'Intermédiaire des Mathematiciens, volume 25, November–December, 1918: "Albert Gauthier-Villars," 121–122; "Questions," 122–126; "Reponses," 127–144.

Journal of the Indian Mathematical Society, volume 10, no. 6, December, 1918: "Infinite series and arithmetical functions" (conclusion) by F. Hallberg, 454–472; "Note on triangles inscribed in an ellipse" by A. N. Raghavachar, 473–475; "On poristic polygons" by F. H. V. Gulasekharam, 475–476; "A geometrical problem" [Given the circumcircle and the orthocenter of a triangle to find the locus of the in- and ex-centers"] by M. T. Naraniengar, 476–478; "In Memoriam" [R. J. Pocock], 479; Problems and solutions, 480–492.

Mathematical Gazette, volume 9, May, 1918: "The elementary theory of statical stability" by S. Brodetsky, 233–236; "Alice through the (convex) looking glass" by W. Garnett, 237–241; "The introduction to infinite series" by W. J. Dobbs, 242–246; "The teachers' library," i–vi. [This last article purports to give "a suggestive list of books suitable for inclusion in the mathematical library of a modern secondary school." In the list are scores of books such as: H. Weber, *Lehrbuch der Algebra*, 3 vols.; P. Muth, *Theorie und Anwendung der Elementarteiler;* E. Pascal, *Gruppi di trasformazioni;* Hamilton, *Elements of quaternions*, 2 vols., and Forsyth, *Theory of differential equations*, 6 vols.]—July: "Alice through the (convex) looking glass" (continued) by W. Garnett, 249–252; "The introduction to infinite series" (continued) by W. J. Dobbs, 253–256; "MacCullagh," 256; Note on the spherical triangle and haversines by E. M. Langley, 257–259; Review by G. Greenhill of D'Ocagne's *Cours de géométrie* (Paris, 1917–18), 261–262—October–December: "Mathematical Association teaching committee, report on the teaching of mechanics" by T. P. Nunn, A. W. Siddons, W. J. Dobbs, and G. Goodwill, 265–292 —January, 1919: "Alice through the (convex) looking glass" (concluded) by W. Garnett, 293–298; "Gleanings far and near" [from *Recollections of Mary Somerville*], 298, 307; "The introduction to infinite series" (concluded) by W. J. Dobbs, 299–301; "A letter from Sir William Rowan Hamilton," 302; "Notes on the life and works of Colin Maclaurin" by C. Tweedie, 303–305; "The equilibrium of jointed frameworks" by G. H. Bryan, 306–307; Review by H. P. Hilton of Darboux's *Principes de géométrie analytique*, 308.

Nouvelles Annales de Mathématiques, volume 77, November, 1918: "Sur les foyers rationnels d'une courbe algébrique plane ou gauche" by P. Appell, 401–402; "Sur les courbes algébriques planes" by R. Bouvaist, 403–417; "Sur deux points du plan d'un triangle et sur une généralisation des points de Brocard" by R. Goormaghtigh, 417–424; "Quelques applications géométriques de la théorie des infiniment petits" by M. Weill, 424–429; "Nouvelles identités" by G. Fontené, 430–431; "Sur la chainette d'égale résistance" by F. Balitrand, 431–433; Solutions and Questions, 434–440.

Quarterly Journal of Pure and Applied Mathematics, volume 48, no. 1, October, 1917: "An expansion related to Stirling's formula, derived by the method of steepest descents"

by G. N. Watson, 1–18; "On parallel curves and evolutes" by A. B. Basset, 19–39; "Some theorems in four-dimensional analysis" by C. E. Weatherburn, 39–58; "The attraction of equiangular spirals" by S. Brodetsky, 58–76; "The normal number of prime factors of a number n" by G. H. Hardy and S. Ramanujan, 76–92; "On the representations of numbers as a sum of $2r$ squares" by L. J. Mordell, 93–96—no. 2, September, 1918: "On the representations of numbers as a sum of $2r$ squares" by L. J. Mordell, 97–104 (conclusion); "On two loci determined by concurrent tangents to plane curves" by C. H. Sisam, 104–112; "Note on asymptotic formulæ for oscillating Dirichlet's integrals" by M. Kuniyeda, 113–135; "Moving axes with variable angles" by E. H. Neville, 136–141; "On a theorem in the theory of differentiation of functions defined by integrals" by G. Fichtenholz, 142–147; "Substitution groups on the terms of symmetric polynomials" by G. A. Miller, 147–150; "On early tables of logarithms and the early history of logarithms" by J. W. L. Glaisher, 151–192.

Revue de Métaphysique et de Morale, volume 26, no. 1, January–February, 1919: "L'entropie, extension conservative" by L. Selme [died January 3, 1919], 89–118; "Gaston Milhaud," 149–151; [Quotation: "Peu d'esprits donnaient au même degré que lui l'impression de la puissance spirituelle, qui s'impose par sa seule lumière, sans prestige matériel et sans contrainte. A toute sa philosophie on pourrait donner pour epigraphe les deux mots qui servaient déjà de conclusion à son premier ouvrage, et qu'il opposait 'aux subtilités d'analyse dont se joue la marche de l'esprit humain': Raison et Liberté"].

Revue générale des Sciences, volume 29, no. 23, December 15, 1918: "Gaston Milhaud" by A. Lalande. [Quotation: "Tous ceux qui l'approchaient étaient attirés et retenus par le charme de sa personne, la finesse de son esprit, la sûreté de son caractère et de ses affections. Jamais philosophe n'a discuté avec plus d'aimable courtoise; et cependant,—ou peut-être par là-même,—il trouvait le moyen de ne rien abandonner de son opinion et souvent de persuader son interlocuteur." Milhaud was born in 1858 and died October 1, 1918. At the age of 20 years he was admitted to both the Ecole polytechnique and the Ecole normale supérieure, but chose the latter and became agrégé ès science mathématique in 1881. He taught in several lycées before he was appointed professor at the University of Paris. He published his *Leçons sur les origines de la science grecque* in 1893. Soon afterwards he became doctor ès lettres at the University of Paris, his principal thesis being *L'essai sur les conditions et les limites de la certitude logique* (1894, 2e éd. 1898). Of his other books the following may be mentioned: *Les philosophes-géomètres de la Grèce: Platon et ses prédécesseurs* (1900); *Le positivisme et le progrès de l'esprit* (1902); *Etudes sur la pensée scientifique chez les Grecs et chez les modernes* (1906); *Nouvelles études sur l'histoire de la pensée scientifique* (1911).]—January 15, 1919: Review by J. Boyer, of Sedgwick and Tyler's *A Short History of Science* (New York, 1918), 28—February 28: "L'avenir du catalogue international de la littérature scientifique," supplément, 17; "Reprise de la publication de la Revue *Isis*" by G. Sarton, supplément, 17.

School Science and Mathematics, volume 19, no. 3, March, 1919: "Proceedings of the eighteenth meeting of the Central Association of Science and Mathematics Teachers," 197–199, 204–208, 265–268; "Progressive science and mathematics courses and teaching in France" by A. Barthelemy, 199–204; "Final report of subcommittee on content of course in first-year mathematics," 259–264; Problems and solutions, 275–281.

Science, volume 49, February 28, 1919: "Common numerals" by G. A. Miller, 215. [Quotation: "It is very interesting to note that during recent years available data relating to the origin of our common number symbols have been carefully reexamined by Carra de Vaux, who published in volume 21 of *Scientia* a brief summary of his results. Among the most surprising of these results are the following: Our common number symbol originated in Europe and from there were transmitted to the Persians. Both India and Arabia received them from Persia, so that the common term *Hindu-Arabic numerals* is decidedly misleading. The common numerals did not come from letters of the alphabet, but were formed directly for the purpose of representing numbers.

It does not appear likely that all of these conclusions reached by Carra de Vaux, who has made an extensive study of intellectual life among the Mohammedans, will be at once accepted, but they tend to exhibit the weak foundation upon which the history of our common numerals has thus far rested."]

Scientia, volume 82, February, 1919: "Sur l'introduction de la méthode des perturbations dans la mécanique générale" by J. M. Plans, 89–93; Reviews, by G. Scorza, of Darboux's *Leçons sur la théorie générale des surfaces*, 2e partie, 2 éd. (Paris, 1915), Darboux's *Principes de géométrie analytique* (Paris, 1917), and of Halphen's *Oeuvres*, tome 1 (Paris, 1916), 147–149.

SCIENTIFIC MONTHLY, volume 8, no. 3, March, 1919: "Charles Edward Pickering" and full-page portrait, 284–286.

SEWANEE REVIEW, volume 27, January–March, 1919: "Edward Kidder Graham" [1876–1918, president of the University of North Carolina], by A. Henderson, 101–106.

TRANSACTIONS OF THE ACADEMY OF SCIENCES OF ST. LOUIS, volume 23; all of the mathematical articles so far published in this volume are by the septuagenarian physicist, F. E. Nipher, author of the little book *Introduction to graphical algebra* published about twenty years ago. No. 4, July, 1916: "Disturbances impressed upon the earth's magnetic field," 153–162; "Gravitation and electrical action," 163–175—No. 5, November, 1917: "Gravitational repulsion," 177–192—No. 6, May, 1918: "Graphical algebra involving functions of the nth degree," 193–204—No. 7, January, 1919: "Graphical algebra," 205–212.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 20, no. 1, January, 1919: Portrait frontispiece and notice of Maxime Bôcher; [This portrait may be obtained by sending twenty cents in postage stamps to the Society, 501 West 116th Street, New York City.] "Necessary conditions in the problems of Mayer in the calculus of variations" by G. A. Larew, 1–22; "Linear equations with unsymmetric systems of coefficients" by Anna J. Pell, 23–39; "On convex functions" by H. Blumberg, 40–44; "Projective transformations in function space" by L. L. Dines, 45–65; "On the order of primitive groups (IV)" by W. A. Manning, 66–78.

AMERICAN DOCTORAL DISSERTATIONS.

G. S. COUNTS, *Arithmetic tests and studies in the psychology of arithmetic* (Supplementary Educational Monographs, vol. 1, no. 4). University of Chicago Press, Chicago, 1917. 4 + 127 pp. (Chicago, 1916.)

G. JAMES, 1882– , *Some theorems on the summation of divergent series.* New York, 227 West 17th St., W. D. Gray, 1917. 28 pp. (Columbia, 1917.)

W. S. MONROE, *Development of arithmetic as a school subject.* (Reprinted from *United States Bureau of Education,* Bulletin, 1917, no. 10.) Washington, D. C., 1917. 170 pp. (Chicago, 1915.)

AGNES L. ROGERS, *Experimental tests of mathematical ability and their prognostic value.* (Teachers College, Columbia University. *Contributions to Education,* no. 89.) New York, Columbia University, 1918. 5 + 118 pp. (Columbia University, 1917.)

C. H. YEATON, 1886– , *Surfaces characterized by certain special properties of their directrix congruences.* (Reprinted from *Annali di matematica,* series 3, vol. 26, 1916.) 3 + 33 pp. (Chicago, 1915.)

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF THE UNIVERSITY OF KANSAS, Lawrence, Kansas. [1918, 35–36, 450–1, 459.]

The officers of the club for the year 1918–19 are as follows: President, Wealthy Babcock '19; vice-president, Faye Dodderidge '19; secretary-treasurer, Edith Whitcher '19; reporter, Viola Engle '19; faculty adviser, Professor Charles H. Ashton; program committee, Rachel Bell '19, Viola Engle '19, Faye Dodderidge '19, Josephine Montague '19.

Below are given the programs for the winter and spring quarters.

December 30, 1918: "Chief contributions of mathematics to astronomy" by Professor Ellis B. Stouffer.

January 13, 1919: "The slide rule" by Evelina Watt '20.
January 27: "Valid aims of teaching mathematics in secondary schools"[1] by Josephine Montague '19.
February 10: "Russian peasant method of calculation" by Viola Engle '19; "Probability curves" by Beatrice Hagen '20.
February 24: "Abridged notation" by Jessie Craig '20.
March 10: "History of the calculus" by Vesta Shafer '19.
March 24: "Trilinear coördinates" by D'Estell Tremaine '19.
April 7: "Method of reciprocal polars" by Marie Brown '19.
April 21: "Origin of logarithms" by Carroll McDowell '19; "Polar planimeter" by Ruth Kelsey '20.
May 5: "Methods of projection" by Hazel Quick '19.
May 19: "Projective reflection" by Nellie Young '19.
June 2: Annual picnic.

THE JUNIOR MATHEMATICS CLUB, University of Minnesota, Minneapolis, Minn. [1918, 312].

The only officers of the club for the current year are an executive committee consisting of Professor Raymond W. Brink, chairman, Miss Ella Thorp, Instructor, and Laura Menk '19.

The programs given so far this year are as follows.

December 12, 1918: "Dimensionality" by Dr. Chester H. Yeaton, Instructor; "Leonhard Euler" by Ruth Stephens Gr.
February 6, 1919: "Gottfried Wilhelm Leibniz" by Lois Huney '19; "A geometrical method of summing a geometrical series'" by Professor William H. Bussey.

Refreshments were served in the faculty parlor after each meeting.

TOPICS FOR CLUB PROGRAMS.

15. THE NUMBER π.

The fact that at least a dozen times in the club programs published during 1918 one finds topics relating to the number π such as "The number π," "Various definitions of π," "History of π," "Squaring the Circle" (which occurs six times), "Quadrature of the Circle," etc., is evidence that investigations of this remarkable number are still as interesting as they were in the time of Archimedes.

Since no member of a mathematics club is a "paradoxer," in De Morgan's sense of the term, little of this interest can be credited to the fact that the problem of the quadrature of the circle is, as De Morgan[2] put it, "connected with one of those propensities, the love of the marvellous, which, carried to an undue extent, tend more than others to throw the mind off its balance, and destroy the comfort

[1] *Cf.* "Valid aims and purposes for the study of mathematics in secondary schools," by A. Davis, *School Science and Mathematics*, vol. XVIII, pp. 112–123, 208–220, 313–324. (Committee Report, Mathematics Club of Chicago.)

[2] In his article "Quadrature of the Circle" in the *Penny Cyclopedia*, vol. 19, p. 186.

of the individual." In fact, the natural interest of mathematicians has undoubtedly been greatly lessened by the activities of "circle-squarers" of the type made famous by De Morgan's witty satire.

The desire for a better understanding of transcendental numbers leads the student of mathematics at the present time to a scientific interest in π, since π was the first transcendental number encountered by humanity (although not the first whose transcendence was proved) and it seems natural to think of it as the simplest and most familiar of the transcendental numbers. That π was first discovered in its relation to the circle is, of course, a mere accident. It was long since pointed out[1] that π is a number which occurs in various natural relations and would enter into analysis from whatever side the subject was approached.

When the ancient Greeks first attempted to find the length of the diagonal of a unit square, they were probably of the opinion that such a fraction (greater than unity, of course) existed, but that they were merely unable to determine it. Undoubtedly many a Greek mathematician had spent much time in fruitless testing of fractions, hoping to find one whose square was 2, before the suspicion arose that no such fraction existed and some one finally succeeded in making the proof to that effect as given by Euclid. Even then they did not conceive of a new number, different in kind from any they knew but no less definite, and invent a symbol to represent it.

Of a somewhat similar nature was the problem of determining the number which should represent the ratio of a circle to its diameter. The problem was, however, much more difficult, for two reasons. In the first place, it was much more difficult to test whether or not a given number was greater or less than the desired number and, in the second place, they were unable to prove that no such rational number existed. It must have been many centuries after the first approximate values were determined before any one suspected that there might be a number, definite and exact, but entirely different in nature from any integer or fraction, which represented the ratio of a circumference to its diameter. It was many more centuries before the first definite information regarding the true nature of π was established when J. H. Lambert, in 1761, communicated to the Berlin Academy an essentially rigorous proof[2] of the irrationality of π. Even

[1] *Cf.* DeMorgan, *Budget of Paradoxes* (London, 1872), pp. 171–172, second ed., edited by D. E. Smith (Chicago, 1915), vol. 1, pp. 284–286; also, Ball, *Mathematical Recreations and Essays*, 4th ed. (London, 1905), pp. 249–50, 5th ed. (London, 1911), p. 295.

[2] The usual citation for Lambert's proof is his "Mémoire sur quelques propriétés remarquables des quantités transcendentes circulaires et logarithmiques," *Mémoires de l'Académie de Berlin* for 1761, Berlin, 1768, pp. 265–322. It is also cited as published in *Beiträge zum Gebrauche der Mathematik*, Bd. II, Berlin, 1770, S. 140–149.

The following paragraph from E. W. Hobson's *Squaring the Circle*, Cambridge, 1913, p. 44, is worthy of note in this connection.

"It has frequently been stated that the first rigorous proof of Lambert's results is due to Legendre (1752–1833), who proved these theorems in his *Eléments de Géométrie* (1794), by the same method, and added a proof that π^2 is an irrational number. The essential rigour of Lambert's proof has however been pointed out by Pringsheim (*Münch. Akad. Ber.*, Kl. 28, 1898), who has supplemented the investigation in respect of the convergence."

before Lambert's proof was given men were coming to believe that π was not only irrational but not an algebraic irrational. This belief was rendered the more probable by Liouville's proof[1] in 1840 of the existence of transcendental numbers, and finally confirmed by Lindemann's proof[2] in 1882 of the transcendence of π.

The literature of the subject is abundant and steadily increasing. For various articles in the periodical literature the reader will, of course, consult the mathematical encyclopedias, the Royal Society Index (especially pp. 233 and 434–436) and the volumes of the International Catalogue. Among the best recent special discussions of the subject are those of Hobson,[3] Young,[4] Beman and Smith,[5] Ball,[6] Enriques,[7] Teixeira,[8] Rudio,[9] Schubert,[10] and Tropfke.[11]

In the books of Beman and Smith, Young and Hobson the development of the subject is considered as falling into three periods, the last named author devoting a chapter to each period.

The first period may be characterized as the *empirical* period, extending from the earliest attempts at the quadrature of the circle to the invention of the calculus in the second half of the seventeenth century. During this period approximations for π are obtained by purely geometrical means until the limit of refinement of that method is reached.

The second period may be characterized as the *analytic* period, extending from the invention of the calculus about a century to the proof of the irrationality of π by Lambert in 1761. During this period, by means of the more powerful methods of the new analysis, π is expressed in terms of infinite products,

[1] The simpler of Liouville's methods is given by Hobson, *loc. cit.*, pp. 44–46.

[2] *Mathematische Annalen*, Bd. 20, S. 220–224, and *Berichte der Berliner Akademie*, 1882, Bd. 2, S. 679–682. Lindemann's proof is based upon Hermite's proof (*Comptes Rendus*, vol. 77, pp. 18–24, 74–79, 226–233, 285–292) of the transcendence of e.

Simplified forms of Hermite's and Lindemann's proofs are given in the books by J. W. A. Young (pp. 402–416) and Beman and Smith (pp. 61–67) cited below, and in various other books readily accessible.

[3] E. W. Hobson, *Squaring the Circle*, Cambridge University Press, 1913.

[4] J. W. A. Young, *Monographs on Modern Mathematics*, New York, 1911, Monograph IX, "The History and Transcendence of π," by D. E. Smith.

[5] W. W. Beman and D. E. Smith, *Famous Problems of Elementary Geometry*, Boston, 1897, English translation of Klein's *Vorträge uber ausgewählte Fragen der Elementar-Geometrie*, Leipzig, 1895, pp. 54–80.

[6] W. W. R. Ball, *Mathematical Recreations and Essays*, 5th ed., London, 1911, pp. 293–306; 4th ed., London, 1905, pp. 247–261.

[7] F. Enriques, *Fragen der Elementargeometrie*, Deutsche Ausgabe von H. Fleischer, II Teil, Leipzig, 1907.

[8] F. G. Teixeira, *Sur les problèmes célèbres de la géométrie élémentaire non résolubles avec la règle et le compas*, Coïmbre, 1915, pp. 83–104.

[9] F. Rudio, *Archimedes, Huygens, Lambert, Legendre. Vier Abhandlungen über die Kreismessung*, Leipzig, 1892. See also articles "Der Bericht des Simplicius uber die Quadraturen des Antiphon und des Hippokrates" and "Zur Rehabilitation des Simplicius" by Rudio in *Bibliotheca Mathematica*, 1902, pp. 7–62, and 1903, pp. 13–18, 1907, pp. 13–22. Also a discussion of Rudio's first article by P. Tannery in the same volume (Bib. Math. for 1902), pp. 342–349.

[10] H. Schubert, *Mathematical Essays and Recreations*, translated by T. J. McCormack, Chicago, 1910, pp. 112–143.

[11] J. Tropfke, *Geschichte der Elementar-Mathematik*, Bd. 2, Leipzig, 1903, S. 108–138.

continued fractions and infinite series. Approximations[1] are obtained far beyond any conceivable practical need and mathematicians come to suspect that π is not only irrational but not an algebraic irrational.

The third period may be called the *critical* period, since it was devoted to "critical investigations of the true nature of the number π itself, considered independently of mere analytical representation." [2] The period extends from the middle of the eighteenth century late into the nineteenth century when the transcendence of π is finally definitely established.

William Jones[3] seems to have been the first to make use of the symbol π with its present special significance, but its permanent use as such was chiefly due to the influence of Euler.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

2773. Proposed by JOSEPH ROSENBAUM, Milford, Conn.

Point out the fallacy in the proof following the problem:

In the triangle $A_1B_1C_1$ let M be a point such that the sum of the distances from it to the sides is a maximum; also, let $A_2B_2C_2$ be a triangle formed by drawing lines through the vertices A_1, B_1, and C_1 parallel to their opposite sides. Then the sum of the distances from M to the sides of the triangle $A_2B_2C_2$ is a minimum.

Proof.—Because the sides of the two triangles are parallel in pairs, the sum of the distances from a variable point P in triangle $A_1B_1C_1$ to the six sides of the two triangles is constant. Now by hypothesis M is a point for which one part of this constant sum is a maximum, and hence it follows that the other part is a minimum.

2774. Proposed by FRANK IRWIN, University of California.

Evaluate the circulants

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & 1 & 2 & \cdots & n-2 & n-1 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 2 & 3 & 4 & \cdots & n & 1 \end{vmatrix}, \qquad \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ a_2 & a_3 & a_4 & \cdots & a_n & a_1 \end{vmatrix},$$

where, in the latter, $a_1, a_2, \cdots a_n$ form an arithmetical progression.

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[1] A condensed table of approximations of π as determined by various men is given by J. W. L. Glaisher, *Messenger of Mathematics*, vol. 2, pp. 122–128, and in vol. 3, pp. 45–46, some corrections are made of the table given on p. 122 of vol. 2. The approximation was carried to 707 places by William Shanks in 1873 (*Proceedings of the Royal Society of London*, vol. 21, p. 318 and vol. 22, p. 45). A considerable list of approximations is given by Ball, *loc. cit.*, 4th ed., pp. 250–261, 5th ed., pp. 296–306.

[2] Hobson, *l. c.*, p. 12.

[3] *Synopsis Palmariorum Matheseos*, London, 1706, pp. 243, 263, *et seq.* Cited by Ball, *loc. cit.*, 4th ed., p. 250, 5th ed., p. 296, and by others. Concerning its early use by others, see article "Sur le premier emploi du symbole π pour 3.14159$\cdots$" by G. Enestrŏm, *Bibliotheca Mathematica*, 1889, p. 28.

2775. Proposed by H. T. BURGESS, University of Wisconsin.

Solve in finite form, if possible, the differential equation

$$\frac{d^2y}{dt^2} + ay\frac{dy}{dt} + by = g,$$

when a and b are arbitrary constants and $g = 32.16$. When $t = 0$, $y = 0$, $dy/dt = 0$.

As solutions have not been received for more than 130 problems proposed since January, 1913, the number of *new* problems proposed each month is to be considerably reduced and *old* problems are to be reproposed. Solutions are desired for the following problems proposed before January, 1918:

Algebra—406, 411, 416, 417, 461, 481, 494.
Geometry—442, 446, 455, 463, 470, 472, 476, 477, 478, 499, 501, 510, 519, 523.
Calculus—348, 349, 353, 406, 415, 429, 432, 434, 436.
Mechanics—287, 291, 300, 308, 309, 313, 315, 332, 343, 344, 351.
Number Theory—198, 201, 202, 205, 231, 232, 234, 238, 245, 247, 263, 266, 270, 272, 273, 274, 275.

430 (Algebra) [March, 1915]. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve the following equations both algebraically and graphically:

$$x^y + y^x = xy, \qquad x^x + y^y = x + y.$$

339 (Calculus) [June, 1913]. Proposed by T. H. GRONWALL, New York, N. Y.

To show that for any real value of x

$$\left|\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)\right| \leqq \frac{1}{n+1}, \quad \text{and} \quad \left|\frac{d^n}{dx^n}\left(\frac{1-\cos x}{x}\right)\right| \leqq \frac{1}{n+1}.$$

340 (Calculus) [June, 1913]. Proposed by C. N. SCHMALL, New York, N. Y.

A pencil of parallel rays of light is incident upon a lens whose faces have the radii r_1, r_2, respectively. Show that the distance of the principal focus from the center of the first face of the lens will be a maximum or a minimum when

$$\frac{r_1}{r_2} = \frac{(\mu-1)^{1/2}}{1+(\mu-1)^{1/2}},$$

where μ has its usual meaning.

272 (Mechanics) [February, 1913]. Proposed by J. F. LAWRENCE, Stillwater, Okla.

A perfectly rough circular cylinder is fixed with its axis horizontal. A sphere is placed on it in a position of unstable equilibrium, and projected with a given velocity parallel to the axis of the cylinder. If the sphere be slightly disturbed in a horizontal direction perpendicular to the direction of the axis of the cylinder, determine at what point the sphere will leave the cylinder.

277[1] (Mechanics) [June, 1913]. Proposed by W. J. GREENSTREET, Editor of the Mathematical Gazette, England.

Around a smooth fixed circular pulley is wound a massless inextensible string, and straight portions go to two free ends A and B to which masses are fastened. The mass at A is initially projected perpendicular to the string while the other is initially at rest. The length of the straight portion to the first mass is initially l and subsequently is r. Find the velocity of the second mass at that moment.

[1] Incorrectly numbered 272 when first proposed.

279[1] **(Mechanics) [September, 1913]. Proposed by W. W. LANDIS, Dickinson College.**

A dam backs up the water for two miles. If the dam is raised 18 inches, will the water two miles up the stream be raised 18 inches, more or less?

191[2] **(Number Theory) [June, 1913]. Proposed by L. E. DICKSON, University of Chicago.**

Find an amicable number triple by solving one of the equations (other than the last) in the MONTHLY, March, 1913, page 92. Note that a solution a is to be excluded if not prime to the numbers in the same line.

192[3] **(Number Theory) [June, 1913]. Proposed by the late ARTEMAS MARTIN.**

Find rational values for v, w, and x that will satisfy simultaneously the conditions

$$(m^2+n^2)(v^2+w^2+x^2)^2 - 4m^2n^2v^2 + m^2n^2(m^2+n^2) = 0,$$
$$(m^2+n^2)(v^2+w^2+x^2)^2 - 4m^2n^2w^2 + m^2n^2(m^2+n^2) = 0,$$
$$(m^2+n^2)(v^2+w^2+x^2)^2 - 4m^2n^2x^2 + m^2n^2(m^2+n^2) = 0,$$

m and n being known quantities.

196[4] **(Number Theory) [September, 1913]. Proposed by CHARLES MACAULAY, Chicago, Ill.**

Combinations containing an even number of letters are formed of the letters a, b, c, d, etc. It is required to place the letters in two columns, so that half the letters in every combination are placed in one column and the other letters of the combination in the other column, and so that all the a's are placed in the same column; all b's in the same column; all the c's in the same column, etc.

SOLUTIONS OF PROBLEMS.

2667 [January, 1918]. Proposed by E. L. REES, University of Kentucky.

Given one diagonal of a parallelogram and the area of the rectangle whose sides are equal to those of the parallelogram, construct the parallelogram so that the diagonal shall make a given angle, α, with a given line and so that the sum of the angles that two adjacent sides make with this line shall be equal to a given angle, β.

2682 [March, 1918]. Proposed by E. L. REES, University of Kentucky.

Given the diagonal and the angle it makes with the bisector of one of the angles of a parallelogram. Construct the parallelogram so that the rectangle having sides equal to those of the parallelogram may have a given area.

SOLUTION BY THE PROPOSER.

If we place one end of the diagonal at the origin in the complex plane and let the fixed line be the real axis, it will be seen at once that what we have given is equivalent to the sum and product of the complex numbers represented by two of the vertices of the required parallelogram. Let this sum and product be denoted by a and b respectively, where mod a = length of diagonal, amp $a = \alpha$, mod b = given area, amp $b = \beta$.

The solution of our problem then requires merely the construction of the complex roots of the quadratic equation $x^2 - ax + b = 0$. This construction is effected by carrying out the operations indicated in the formula $x = a/2 \pm \sqrt{[(a/2)^2 - b]}$ all of which are possible with ruler and compasses.

It will be noted that problem no. 2682 is a special case of no. 2667 and hence the method of solution here suggested is applicable also to it.

Also solved by H. N. CARLETON.

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[1] Incorrectly numbered 274 when first proposed.
[2] Incorrectly numbered 187 when first proposed.
[3] Incorrectly numbered 188 when first proposed.
[4] Incorrectly numbered 192 when first proposed.

2700 [May, 1918]. Proposed by the late ARTEMAS MARTIN.

In a factory 250 men are paid an average wage of \$15 each per week. The men are paid unequally, the wages being \$20, \$16, \$10, and \$8 per week, respectively, for different classes of work. How many are employed at each rate of pay?

NOTE.—I am told that this question was set in a Civil Service examination paper to be worked by arithmetic. 2,896 answers have been found. Are there any more?

SOLUTION BY H. S. UHLER, Yale University.

Let w, x, y, and z denote the number of men receiving \$20, \$16, \$10, and \$8 per week, respectively. The conditions of the problem lead at once to the following equations:

$$w + x + y + z = 250,$$

$$(20w + 16x + 10y + 8z)/250 = 15.$$

Elimination of z gives

$$6w + 4x + y = 875.$$

The last equation must be solved for positive integral values (zero included) of w, x, and y. This may be accomplished by assigning to w the values 0, 1, 2, 3, $\cdots$ and then discussing the number of possible solutions conditioned by the limited arithmetical progressions involving x, y, and z.

Since $y = 875 - 6w - 4x$ it is evident that the greatest value of w which will make y positive is 145. Let w' symbolize any number in the sequence 0, 1, 2, 3, $\cdots$, 144, 145. We may now imagine the following table filled out numerically:

w	x	y	z
w'	x'	$875 - 6w' - 4x'$	$5w' - 625 + 3x'$
w'	$x' - 1$	$875 - 6w' - 4x' + 4$	$5w' - 625 + 3x' - 3$
w'	$x' - 2$	$875 - 6w' - 4x' + 8$	$5w' - 625 + 3x' - 6$
.	. .	. . . . .	. . . . .

For a given value of w (w') the number of possible solutions is equal to the greatest number of rows that can be written in the above schematic table without introducing a negative value in one or more of the last three columns.

CASE 1.—*w' odd.* Let $w' = 2k + 1$, then

$$y = 1 + 4(217 - 3k - x'),$$

hence the greatest integral value of x' which will make y positive is $217 - 3k$. Accordingly, when w' is odd, the first or top row of the table will consist of the elements $2k + 1$, $217 - 3k$, 1, and $k + 31$ under the headings w, x, y, and z, respectively. Since the greatest value of w' is 145 the corresponding value of k equals 72, hence the least value of $217 - 3k$ is 1 so that all the elements of the above first row are positive.

Attention will now be directed to the last or bottom row of the table. When the total number of rows is $218 - 3k$ the elements of the last row will be $2k + 1$, 0, $869 - 12k$, and $10(k - 62)$, for the common differences of the second, third, and fourth columns are -1, 4, and -3, respectively. Therefore, as long as k does not fall below 62 the second column will limit the number of rows in the table. Under the conditions that w' be odd and $k \geqq 62$ the number of rows in a table equals $218 - 3k$ so that the number of solutions of the problem is the sum $2 + 5 + \cdots + 29 + 32 = 187$.

For values of k less than 62 the fourth column will limit the number or rows in the table, instead of the second column. Let k be of the form $3m + 2$, then the nth term of the fourth column may be written $z_n = 3(12 + m - n)$. This will vanish whenever $n = 12 + m$. Positive solutions for k, equal to $3m + 2$, and less than 62, are obtained when $m = 19, 18, \cdots$, or $k = 59, 56, \cdots$, or $w' = 119, 113, \cdots$. When $k = 3m + 3$ and $k = 3m + 4$ the $(12 + m)$th terms of the fourth column will be 1 and 2, respectively. On the other hand, the $(13 + m)$th terms will be -2 and -1, in the same order. Consequently, for odd values of w', corresponding to values of k less than 62, the tables may be collected in groups of three having the same number of rows or possible solutions. The number of rows is evidently $m + 12$. If we allow m to decrease from 19 to -1, or k from 61, $(3m + 4)$, to -1, $(3m + 2)$, we shall obtain not only

the number of possible solutions pertaining to values of w' from 123 to 1 but also one inadmissible solution belonging to $w' = -1$. Consequently, the number of possible solutions associated with odd values of w' and not included in the 187 given above is equal to $3(31 + 30 + \cdots + 12 + 11) - 11 = 1312$. Finally, the total number of possible solutions of the problem, when w' is odd, equals $187 + 1312 = 1499$.

CASE 2.—*w' even.* Let $w' = 2k$, then

$$y = 3 + 4(218 - 3k - x'),$$

hence $218 - 3k$ is the greatest integral value of x' that will make y positive. Accordingly, when w' is even, the first row of the table will comprise the constituents $2k$, $218 - 3k$, 3, and $29 + k$ under the headings w, x, y, and z, respectively. Since the greatest even value of w' is 144 the corresponding value of k equals 72, hence the least value of $218 - 3k$ is 2 so that all the elements of the above first row are positive.

When the total number of rows is $219 - 3k$ the elements of the last row will be $2k$, 0, $875 - 12k$, and $10k - 625$. Therefore, as long as k is not less than 63 the second column will limit the number of rows in the table. Under the present conditions the number of rows in a table equals $219 - 3k$ so that the number of solutions of the problem is the sum

$$3 + 6 + \cdots + 27 + 30 = 165.$$

For values of k less than 63 the fourth column will limit the number of rows in the table. Let k be of the form $3m + 1$, then the nth term of the fourth column may be written $z_n = 3(11 + m - n)$. This will vanish whenever $n = 11 + m$. Positive solutions for k, equal to $3m + 1$, and less than 63, are obtained when $m = 20, 19, \cdots$, or $k = 61, 58, \cdots$, or $w' = 122, 116, \cdots$. When $k = 3m + 2$ and $k = 3m + 3$ the $(11 + m)$th terms of the fourth column will be 1 and 2 respectively. On the other hand, the $(12 + m)$th terms will be -2 and -1, in the order named. Consequently, for even values of w', corresponding to values of k less than 63, the tables may be collected in groups of three having the same number of rows or possible solutions. The number of rows is obviously $m + 11$. If we allow m to decrease from 20 to 0, or k from 63, $(3m + 3)$, to 1, $(3m + 1)$, we shall include the case of $w' = 126$ which has already been disposed of and omit the number of solutions for $w' = 0$, namely 10. Consequently, the number of solutions not already accounted for equals $3(31 + 30 + \cdots + 12 + 11) - 31 + 10 = 1302$. Finally, the total number of possible solutions of the problem, when w' is even (or zero), equals $165 + 1302 = 1467$. Therefore, for all admissible values of w', the complete number of solutions is 2966. Hence 70 more solutions exist than the proposer states have already been found.

REMARK.—The number of solutions involving one or more zeros is 72.

N = number of solutions in the table.
N' = number of rows containing one or more zeros.
Only the first and last rows are given for each value of w.

w	x	y	z	N	N'	w	x	y	z	N	N'	w	x	y	z	N	N'
145	1	1	103	2	1	137	13	1	99	14	1	129	25	1	95	26	1
145	0	5	100			137	0	53	60			129	0	101	20		
144	2	3	101	3	1	136	14	3	97	15	1	128	26	3	93	27	1
144	0	11	95			136	0	59	55			128	0	107	15		
143	4	1	102	5	1	135	16	1	98	17	1	127	28	1	94	29	1
143	0	17	90			135	0	65	50			127	0	113	10		
142	5	3	100	6	1	134	17	3	96	18	1	126	29	3	92	30	1
142	0	23	85			134	0	71	45			126	0	119	5		
141	7	1	101	8	1	133	19	1	97	20	1	125	31	1	93	32	1
141	0	29	80			133	0	77	40			125	0	125	0		
140	8	3	99	9	1	132	20	3	95	21	1	124	32	3	91	31	0
140	0	35	75			132	0	83	35			124	2	123	1		
139	10	1	100	11	1	131	22	1	96	23	1	123	34	1	92	31	0
139	0	41	70			131	0	89	30			123	4	121	2		
138	11	3	98	12	1	130	23	3	94	24	1	122	35	3	90	31	1
138	0	47	65			130	0	95	25			122	5	123	0		

w	x	y	z	N	N'	w	x	y	z	N	N'	w	x	y	z	N	N'
121	37	1	91	31	0	96	74	3	77	26	0	71	112	1	66	23	1
121	7	121	1			96	49	103	2			71	90	89	0		
120	38	3	89	30	0	95	76	1	78	27	1	70	113	3	64	22	0
120	9	119	2			95	50	105	0			70	92	87	1		
119	40	1	90	31	1	94	77	3	76	26	0	69	115	1	65	22	0
119	10	121	0			94	52	103	1			69	94	85	2		
118	41	3	88	30	0	93	79	1	77	26	0	68	116	3	63	22	1
118	12	119	1			93	54	101	2			68	95	87	0		
117	43	1	89	30	0	92	80	3	75	26	1	67	118	1	64	22	0
117	14	117	2			92	55	103	0			67	97	85	1		
116	44	3	87	30	1	91	82	1	76	26	0	66	119	3	62	21	0
116	15	119	0			91	57	101	1			66	99	83	2		
115	46	1	88	30	0	90	83	3	74	25	0	65	121	1	63	22	1
115	17	117	1			90	59	99	2			65	100	85	0		
114	47	3	86	29	0	89	85	1	75	26	1	64	122	3	61	21	0
114	19	115	2			89	60	101	0			64	102	83	1		
113	49	1	87	30	1	88	86	3	73	25	0	63	124	1	62	21	0
113	20	117	0			88	62	99	1			63	104	81	2		
112	50	3	85	29	0	87	88	1	74	25	0	62	125	3	60	21	1
112	22	115	1			87	64	97	2			62	105	83	0		
111	52	1	86	29	0	86	89	3	72	25	1	61	127	1	61	21	0
111	24	113	2			86	65	99	0			61	107	81	1		
110	53	3	84	29	1	85	91	1	73	25	0	60	128	3	59	20	0
110	25	115	0			85	67	97	1			60	109	79	2		
109	55	1	85	29	0	84	92	3	71	24	0	59	130	1	60	21	1
109	27	113	1			84	69	95	2			59	110	81	0		
108	56	3	83	28	0	83	94	1	72	25	1	58	131	3	58	20	0
108	29	111	2			83	70	97	0			58	112	79	1		
107	58	1	84	29	1	82	95	3	70	24	0	57	133	1	59	20	0
107	30	113	0			82	72	95	1			57	114	77	2		
106	59	3	82	28	0	81	97	1	71	24	0	56	134	3	57	20	1
106	32	111	1			81	74	93	2			56	115	79	0		
105	61	1	83	28	0	80	98	3	69	24	1	55	136	1	58	20	0
105	34	109	2			80	75	95	0			55	117	77	1		
104	62	3	81	28	1	79	100	1	70	24	0	54	137	3	56	19	0
104	35	111	0			79	77	93	1			54	119	75	2		
103	64	1	82	28	0	78	101	3	68	23	0	53	139	1	57	20	1
103	37	109	1			78	79	91	2			53	120	77	0		
102	65	3	80	27	0	77	103	1	69	24	1	52	140	3	55	19	0
102	39	107	2			77	80	93	0			52	122	75	1		
101	67	1	81	28	1	76	104	3	67	23	0	51	142	1	56	19	0
101	40	109	0			76	82	91	1			51	124	73	2		
100	68	3	79	27	0	75	106	1	68	23	0	50	143	3	54	19	1
100	42	107	1			75	84	89	2			50	125	75	0		
99	70	1	80	27	0	74	107	3	66	23	1	49	145	1	55	19	0
99	44	105	2			74	85	91	0			49	127	73	1		
98	71	3	78	27	1	73	109	1	67	23	0	48	146	3	53	18	0
98	45	107	0			73	87	89	1			48	129	71	2		
97	73	1	79	27	0	72	110	3	65	22	0	47	148	1	54	19	1
97	47	105	1			72	89	87	2			47	130	73	0		

w	x	y	z	N	N'	w	x	y	z	N	N'	w	x	y	z	N	N'
46	149	3	52	18	0	30	173	3	44	15	0	14	197	3	36	13	1
46	132	71	1			30	159	59	2			14	185	51	0		
45	151	1	53	18	0	29	175	1	45	16	1	13	199	1	37	13	0
45	134	69	2			29	160	61	0			13	187	49	1		
44	152	3	51	18	1	28	176	3	43	15	0	12	200	3	35	12	0
44	135	71	0			28	162	59	1			12	189	47	2		
43	154	1	52	18	0	27	178	1	44	15	0	11	202	1	36	13	1
43	137	69	1			27	164	57	2			11	190	49	0		
42	155	3	50	17	0	26	179	3	42	15	1	10	203	3	34	12	0
42	139	67	2			26	165	59	0			10	192	47	1		
41	157	1	51	18	1	25	181	1	43	15	0	9	205	1	35	12	0
41	140	69	0			25	167	57	1			9	194	45	2		
40	158	3	49	17	0	24	182	3	41	14	0	8	206	3	33	12	1
40	142	67	1			24	169	55	2			8	195	47	0		
39	160	1	50	17	0	23	184	1	42	15	1	7	208	1	34	12	0
39	144	65	2			23	170	57	0			7	197	45	1		
38	161	3	48	17	1	22	185	3	40	14	0	6	209	3	32	11	0
38	145	67	0			22	172	55	1			6	199	43	2		
37	163	1	49	17	0	21	187	1	41	14	0	5	211	1	33	12	1
37	147	65	1			21	174	53	2			5	200	45	0		
36	164	3	47	16	0	20	188	3	39	14	1	4	212	3	31	11	0
36	149	63	2			20	175	55	0			4	202	43	1		
35	166	1	48	17	1	19	190	1	40	14	0	3	214	1	32	11	0
35	150	65	0			19	177	53	1			3	204	41	2		
34	167	3	46	16	0	18	191	3	38	13	0	2	215	3	30	11	1
34	152	63	1			18	179	51	2			2	205	43	0		
33	169	1	47	16	0	17	193	1	39	14	1	1	217	1	31	11	0
33	154	61	2			17	180	53	0			1	207	41	1		
32	170	3	45	16	1	16	194	3	37	13	0	0	218	3	29	10	10
32	155	63	0			16	182	51	1			0	209	39	2		
31	172	1	46	16	0	15	196	1	38	13	0						
31	157	61	1			15	184	49	2								

Also solved by L. P. SHIDY, R. A. JOHNSON, F. H. LOUD, H. N. CARLETON, B. F. YANNEY.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

Mr. WARREN WEAVER has been appointed instructor in mathematics at the University of Wisconsin.

Mr. P. A. FRALEIGH, of Cornell University, has been appointed instructor in mathematics at Dartmouth College.

Professor O. D. KELLOGG, of the University of Missouri, has been appointed lecturer in Harvard University for the year 1919–20.

Dr. W. C. GRAUSTEIN, who has given up his work at the Aberdeen Proving Ground, and who has been teaching in Harvard University this spring, has been promoted to be an assistant professor of mathematics.

Professor W. H. GARRETT, of Baker University, is on leave of absence for this Spring, his duties being assumed by Mr. F. E. WOOD, who has been in the national service until quite recently.

Dr. C. E. WILDER, who is now at Clark University, has been appointed assistant professor of mathematics at Northwestern University.

At Wellesley College, Miss HELEN BARTON, now in charge of the department of Science and Mathematics at Salem College, and Miss MARION ELIZABETH STARK, head of the department of Mathematics at Meredith College, have been appointed instructors in mathematics.

At the U. S. Naval Academy at Annapolis the following promotions in the Department of Mathematics are announced as going into effect on April 1: to be Professor, Mr. PAUL CAPRON; to be Associate Professors, Mr. W. J. KING and Mr. J. B. EPPES; to be Assistant Professors, Mr. J. A. BULLARD, Mr. JOHN TYLER, Mr. ARTHUR KIERNAN, Mr. J. N. GALLOWAY, Mr. ALEXANDER DILLINGHAM, and Dr. G. R. CLEMENTS.

Many mathematicians who have called on Mr. ROBERT BOWES, of Bowes and Bowes, at No. 1 Trinity Street, Cambridge (the home of a succession of booksellers for more than four hundred years) will learn with deep regret of his death on February 9, 1919, in the eighty-fourth year of his age. About a year before, the University of Cambridge conferred on him the honorary degree of Master in Arts.

Mr. I. ROMAN, former instructor at Northwestern University, has received his discharge at the Aberdeen Proving Ground, and is studying this Spring at the University of Chicago.

Professor H. F. BLICHFELDT, of Leland Standford University, and Professor H. H. Mitchell, of the University of Pennsylvania, have completed their work at the Aberdeen Proving Ground and returned to their former duties.

Major OSWALD VEBLEN was abroad studying ordnance problems and methods in England, France and Italy, from November to March. He hopes to return to his University work at Princeton in the near future.

Major F. R. MOULTON has completed his work in the Ordnance Department at Washington and received his discharge. He has resumed his duties as professor of astronomy at the University of Chicago this Spring Quarter.

Major J. L. COOLIDGE of Harvard University is reported acting as a liaison officer in the office of one of the French Commissioners in Paris.

On March 1, WILLIAM BURNSIDE retired from his professorship of mathematics at the Royal Naval College, Greenwich. Fellow of Pembroke, second wrangler, first Smith prizeman, Royal Society medallist, and De Morgan medallist, he is known in America more particularly through his work on the *Theory of groups of finite order* (Cambridge, 1897; 2d ed., 1911).

Dr. H. S. WASHINGTON, of the geophysical laboratory, Carnegie Institution, has been elected a foreign member of the Accademia dei Lincei.

Science announces that JACQUES HADAMARD, professor in the Collège de France, has accepted an invitation from Yale University to be a Silliman Lecturer in the Spring of 1920. Professor Hadamard received the honorary degree of LL.D. at the Yale Bicentennial in 1901.

The Adams prize, of value £250, has been awarded to J. W. NICHOLSON, professor of mathematics at King's College, University of London. This prize, "awarded every two years for an essay on some branch of pure mathematics, astronomy, or other branch of natural philosophy" was founded by members of St. John's College, Cambridge, in 1848 in memory of "the first among mathematicians of Europe to determine from perturbations the unknown place of a disturbing planet exterior to Uranus." In 1907 the award was made to Professor E. W. BROWN. Earlier awards were made to J. C. MAXWELL, I. TODHUNTER and E. J. ROUTH.

Nature reports that at Cambridge University, the senate has approved of provision being made for the establishment of the degree of doctor of philosophy. "The syndicate dealing with this question recommends that, subject to certain exemptions, candidates for the degree, before submitting a dissertation, must have pursued a course of research for not less than three years, and the senate has determined that of this period one year, in the case of a graduate of the university, and two years, in the case of other students, must be spent in Cambridge."

In a recent letter to Professor H. E. SLAUGHT, Professor E. R. HEDRICK tells something of his new work in France in helping to organize educational work for the American Expeditionary Forces. The Army Educational Commission (Professors ERSKINE, BUTTERFIELD, and SPAULDING) acting with the new Army Educational Division of general headquarters under General REES (who was in charge of the American S. A. T. C.) has undertaken the establishment at Beaune, in the immense U. S. hospital buildings (with beds for 15,000) a real university to be constituted under the direction of American Heads of Departments with instructors drawn from officers of the A. E. F., and students (all high school

graduates) from the enlisted personnel of the A. E. F. The university was expected to open with 5,000 students and eventually to enroll nearly 20,000 students. About a hundred instructors in mathematics will be needed. Professor Hedrick reported the following as already available as instructors: Mr. JOSEPH ALLEN, Professor H. E. BUCHANAN, Major W. L. HART, Captain REHM, Captain CALDWELL, Captain FENTRESS, Captain SCHNAPP, Lieutenant NELSON, Lieutenant DILLON, and Miss MAY H. VANN.

Secretary of the Navy Daniels has approved an order establishing the usual collegiate grades of instructor, assistant professor, associate professor, and professor for the civilian teachers at the United States Naval Academy, with a schedule of salaries and promotions. The initial salaries for these grades are $2,000, $2,400, $3,000, and $3,600 per annum, respectively, with an automatic increase of $100 per year for each of the two lower grades, and of 10 per cent. of the initial pay of the grade for each five years of service at each of the two higher grades, with maxima of $2,400, $3,000, $3,600, and $4,500 respectively. All new appointments are to be at grade instructor, with promotion by selection to the next higher grade, subject to a requirement of two, five, and ten years respectively of continuous service at the Naval Academy for eligibility to the professorial grades, and subject to the further restriction that for each ten men or major part thereof in any department, there shall not be more than two assistant professors, two associate professors and one professor.

While this scheme does not have the force of statute, the civilian teachers at the Naval Academy being paid from a lump sum appropriation to be expended subject to the approval of the Secretary of the Navy, it is not anticipated that there will be any recessions from the provisions of this plan. There are several instructorships to be filled for the coming year. A competitive examination for the selection of these men (age, between 28 and 40 years) will be given at Annapolis, Md., June 10–12. Inquiries concerning the details of this examination should be addressed to Professor D. M. Garrison, head of the department of mathematics, U. S. Naval Academy.

In addition to the reports of summer courses in mathematics given in the March and April numbers of the MONTHLY, we have the following:

Northwestern University: June 23–August 15. Professor E. J. MOULTON: Differential calculus; Plane trigonometry. Mr. F. L. KERR: Plane analytic geometry; College algebra. Each course bears a credit of three semester-hours.

Cornell University: regular summer session, July 7–August 15. Professor VIRGIL SNYDER: Advanced geometry, credit 5 hours; Solid geometry, 3 hours. Professor D. C. GILLESPIE: Higher analysis, 5 hours; Algebra, 3 hours. Professor W. A. HURWITZ: Differential equations of physics, 3 hours; Analytics and a part of differential calculus, 5 hours. Professor CARVER: Trigonometry, 3 hours. Special session on account of the war, July 7–August 29. Professors F. R. SHARPE, C. E. CRAIG and F. W. OWENS: Integral calculus, 3 hours; Professor

OWENS: Advanced calculus III, 3 hours. Mrs. HELEN B. OWENS: Differential equations III, 3 hours. The last two courses are completions of those begun in January for a year's work.

University of Colorado: First term, June 30th to August 2d. Second term, August 4th to September 6th. *Instructors:* Associate Professor ABRAHAM COHEN, Johns Hopkins University; Assistant Professor LIGHT and Miss CLARIBEL KENDAL, University of Colorado. *Courses:* (1) Solid geometry (first term); (2) Trigonometry (throughout quarter); (3) College algebra (throughout quarter); (4) Plane analytic geometry (throughout quarter); (5) Differential calculus (throughout quarter); (6) History of mathematics (first term); (7) Fundamental concepts of mathematics (first term); (8) Differential equations (first term); (9) Least squares (first term); (10) Calculus of variations (first term); (11–16) Other advanced courses in mathematics. (*a*) Theory of algebraic equations, (*b*) Definite integrals, (*c*) Theory of a complex variable, (*d*) Elliptic integrals and functions, (*e*) An introductory course in analysis, (*f*) Differential geometry (first or second term).

University of Texas: First term, June 10–July 22. Professor H. Y. BENEDICT: The subject matter and teaching of High School mathematics; Algebra. Professor RICE: Solid analytic geometry; Analytic geometry. Professor E. L. DODD: Advanced statistics; Mathematics of investment. Dr. GOLDIE P. HORTON: Calculus; Introduction to analytic geometry. Miss M. E. DECHERD: Plane trigonometry; Solid geometry. Second term, July 22–August 30. Professor M. B. PORTER: Fundamentals in elementary mathematics; Calculus. Professor OGLESBY (of Williams and Mary College): Analytic geometry; Introduction to analytic geometry. Professor H. J. ETTLINGER: Calculus; Plane trigonometry.

Harvard University: There will be two sessions this year of the summer school of arts and sciences, the first extending from July 1 to August 9 and the second from August 11 to September 13. Separate courses in trigonometry and analytic geometry will be given during each session, during the first by Professor C. L. BOUTON and during the second by Professor G. D. BIRKHOFF. Professor BIRKHOFF will give a course beginning the differential and integral calculus during the first session, provided a sufficient number of students sent their names to Professor W. F. OSGOOD, chairman of the division of mathematics, before May 1, and subject to the same provision he will give, also during the first session, a slightly more advanced course in the calculus, presupposing a knowledge of the introductory portions of the calculus. Advanced work in mathematics for graduate students is offered during both sessions, during the first by Professor BOUTON and during the second by Professor BIRKHOFF. Each of the courses mentioned, carried satisfactorily for one session, counts as a half-course for a degree.

Columbia University: July 7–August 15. A. Graduate courses. Professor JAMES MACLAY Geometric constructions. Professor EDWARD KASNER: Graphical methods including nomography and applications of the calculus. Professor W. B. FITE: Functions of a complex variable. B. Undergraduate courses.

Professors FITE, MACLAY and L. P. SICELOFF, and Doctors C. A. FISCHER, LAMSON, G. W. MULLINS and J. F. RITT, and Mr. POST: Differential and integral calculus, Analytics, College algebra, Trigonometry, Plane and solid geometry, and Elementary algebra. Each graduate course is counted as 3 points credit, each undergraduate course as 2 points. In addition to the above, ten courses are offered in Teachers College on the teaching of mathematics and on applied mathematics by Professor C. B. UPTON, Mr. W. E. BRECKENRIDGE and Mr. W. S. SCHLAUCH.

At the April meeting of the National Academy of Sciences Major OSWALD VEBLEN and Professor EDWIN BIDWELL WILSON were elected Fellows.

At the celebration, on September 21, 1918, of the 250th anniversary of the founding of the University of Lund, degrees of doctor *honoris causa* were conferred on two mathematicians, Gustaf Eneström, the editor of *Bibliotheca Mathematica*, and J. L. W. V. JENSEN, telephone-engineer of Copenhagen.

The Rumford Committee of the American Academy of Arts and Sciences has voted the sum of $500.00 to Professor A. G. WEBSTER of Clark University, in aid of his researches in pyrodynamics and practical interior ballistics.

The Academy of Sciences of the Institute of France announces the award of the following prizes for the year 1918: Poncelet prize of 2000 francs to J. LARMOR, of Cambridge University, for the totality of his mathematical researches; the Francoeur prize of 1000 francs to P. MONTEL, of the University of Paris, for his work on sequences of analytic functions; a prize of 2000 francs to the late SAMUEL LATTÈS, professor at the University of Toulouse, for his work in mathematical analysis; a prize of 2000 francs to PAUL BARBARIN, professor at Lycée Saint Louis, Paris, for his work in non-euclidean geometry; a prize of 1500 francs to LOUIS FABRY, assistant astronomer at the Observatory of Marseilles, for his ephemerides of the minor planets; and a prize of 3000 francs to GASTON JULIA, editor of Bôcher's Sorbonne lectures, for a study of iteration submitted to the Academy in competition with two other memoirs, one of which, by the late Samuel Lattès, received honorable mention.

The twelfth regular meeting of the American Mathematical Society at Chicago was held on Friday and Saturday, March 28 and 29. About forty persons attended the meetings including thirty-one members of the Society. On Friday and Saturday mornings there were presented fifteen papers of the usual research nature. The Friday afternoon session was devoted to a symposium on the geometry of numbers. Professors H. F. BLICHFELDT and L. E. DICKSON presented the subject, basing their discussion largely on the works of Minkowski. On Friday evening twenty-nine members took dinner together at the Quadrangle Club. In the after-dinner informal speeches, Professor ALEXANDER ZIWET, of the University of Michigan, gave a most cordial invitation to the mathematical

public there represented to attend the summer meeting of the American Mathematical Society at Ann Arbor, Tuesday–Thursday, September 2–4, 1919. A more complete report of the meeting will be found in the *Bulletin of the American Mathematical Society*. President Morley appointed as members of the committee on arrangements for the summer meeting Professors BEMAN (chairman), BLISS, KARPINSKI, OSGOOD, and the Secretary.

The Association will hold its summer meeting by invitation at the University of Michigan on Thursday–Saturday, September 4–6, 1919, following and in conjunction with the summer meeting of the American Mathematical Society. President Slaught has appointed Professor W. B. FORD chairman of the Program Committee and Professor W. W. BEMAN chairman of the Committee on Local Arrangements. It is expected that a more detailed notice will be published in the June MONTHLY and that the final statement of the program will be mailed to members of the Association about the middle of August.

The finance committee of the Council of the Association has invested five hundred dollars of the reserve funds in a Victory Loan bond; this is in addition to a like amount invested in a Liberty Bond a year and a half ago, as reported at the annual meeting in December, 1917.

The University of Colorado has just announced that instead of the usual summer session of six weeks it will conduct, in the summer of 1919, a Summer Quarter of two terms, greatly increasing the scope and variety of courses. The first term will open on June 30 and close on August 2. The second term will open on August 4 and close on September 6.

Special Notice to All Members of the Association.

The Annual Register of the Association will be published early in the fall, containing such data as were listed in the Register issued in May, 1918. In order to make this list as serviceable as possible for the academic year 1919–1920, members are asked to assure its accuracy and completeness by sending to the Secretary as early as possible notice of any promotions, changes of address, etc., **affecting the next academic year.** All such additions and corrections will be used up to the time of printing the Register, but these should be in hand by September first at the latest. It is hoped that the usefulness of the Annual Register will be greatly enchanced by this plan, which is intended to make it a reliable guide for *the whole academic year*.

Your Vacation Opportunity

The Summer Quarter 1919 will receive the added inspiration of professors and instructors returning from war service in many lands. Students and teachers, interested in keeping abreast of the times or in completing work already begun, appreciate the opportunity of instruction in a regular season of study under members of the University staff. Scholars desiring to prosecute research in the libraries and laboratories will find facilities for work under the most favorable conditions.

Courses are offered in all departments and include undergraduate and graduate instruction in **Arts, Literature, Science, Commerce and Administration, Law, Medicine, Education, and Divinity.**

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Mathematics: See list of courses in News Column of this issue.

One fee for all courses, $15, except Law (10 weeks) $25.

For further announcements address

Registrar, University, Madison, Wisconsin

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There are other journals suited to the Secondary field, and there are still others of technical scientific character in the University field: but the MONTHLY is the only journal of Collegiate grade in America suited to the needs of the non-specialist in mathematics.

THE MATHEMATICAL ASSOCIATION OF AMERICA now has over eleven hundred individual and institutional members. There are already nine sections formed, representing twelve different states. The Association has held so far two national meetings per year, one in September and one in December. The sections, for the most part, hold two meetings each year. All meetings, both national and sectional, are reported in the Official Journal, and many of the papers presented at these meetings are published in full.

The slogan of the Association is to include in its membership every teacher of collegiate mathematics in America and to make such membership worth while. Application blanks for membership may be obtained from the Secretary, W. D. Cairns, 27 King Street, Oberlin, Ohio.

SECOND REPORT[1] OF THE COMMITTEE ON LIBRARIES—A LIST OF COLLEGE MATHEMATICAL TEXT-BOOKS.

Preliminary Statement. At the meeting of the Association held in Chicago in December, 1917, the Library Committee was requested to prepare a list of mathematical text-books used in colleges, thus bringing together in compact and accessible form the bulk of the available information pertaining to this field of instruction. The Committee, in following out the request, began by ascertaining the names and addresses of the various publishing houses of America, consulting for this purpose the so-called *Publisher's Trade List Annual.* A letter was then sent to each house asking for a full list of their mathematical texts of college grade. Replies were eventually received in nearly every instance, and the findings of the Committee as thus determined constitute the basis of the subjoined list. Books published independently, as by an author himself, have also been included whenever coming to the attention of the Committee.

It should be said that some omissions and errors doubtless occur, as would be the case whatever plan of procedure might be followed in attempting to prepare such a list, but since nearly every book listed has been personally inspected, it is hoped that errors are few. It is possible that a supplementary list which shall take account of omissions may be published later. The Chairman of the Committee will be glad to receive any information tending to make the report more accurate and more complete. It may be well to note that the list pertains only to text-books (as distinguished from books of reference) by American authors (or translators), and published in America, since even with such restrictions the list is of considerable length.

FLORIAN CAJORI, University of California;
E. S. CRAWLEY, University of Pennsylvania;
SOLOMON LEFSCHETZ, University of Kansas;
W. R. LONGLEY, Yale University;
R. E. ROOT, U. S. Naval Academy;
W. B. FORD, University of Michigan, *Chairman.*

COLLEGE ALGEBRA.

A. FIRST COURSE.

ASHTON (C. H.) and MARSH (W. R.) *College Algebra.* Scribner, 1907.[2] 9 + 279 pp. $1.50.
BOWSER (E. A.) *College Algebra.* Heath, 1890. 18 + 540 pp. $1.60.
BRENKE (W. C.) *Advanced Algebra.* Century, 1917. 7 + 196 pp. $1.25.
COLLINS (J. V.) *Advanced Algebra.* American Book, 1913. 12mo. 10 + 342 pp. $1.00.
CRATHORNE (A. R.) See RIETZ (H. L.).
DAVISSON (S. C.) *College Algebra.* Macmillan, 1910. 12mo. 16 + 243 pp. $1.50.
DICKSON (L. E.) *College Algebra.* Wiley, 1902. 12mo. 7 + 214 pp. $1.50.
DOWNEY (J. F.) *Higher Algebra.* American Book, 1901. 12mo. 416 pp. $1.50.
FINE (H. B.) *College Algebra.* Ginn, 1904. 12mo. 8 + 595 pp. $2.00.
FISHER (G. E.) and SCHWATT (I.) *Higher Algebra.* Macmillan. 12mo. 615 pp. $2.00.
FITE (W. B.) *College Algebra.* Revised edition. Heath, 1918. 5 + 320 pp. $1.56.

[1] Cf. this MONTHLY, October, 1917.

[2] It is intended that here, and in the following titles, the date shall be that of the latest *edition* but not of the latest *reprint.*